

Clustered Planarity with Pipes^{*}

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Abstract

We study the version of the C-PLANARITY problem in which edges connecting the same pair of clusters must be grouped into pipes, which generalizes the STRIP PLANARITY problem. We give algorithms to decide several families of instances for the two variants in which the order of the pipes around each cluster is given as part of the input or can be chosen by the algorithm.

1 Introduction

Visualizing clustered graphs is a challenging task with several applications in the analysis of networks that exhibit a hierarchical structure. The most established criterion for a readable visualization of these graphs has been formalized in the notion of *c-planarity*, introduced by Feng, Cohen, and Eades [13] in 1995. Given a *clustered graph* $\mathcal{C}(G, \mathcal{T})$ (*c-graph*), that is, a graph G equipped with a recursive clustering \mathcal{T} of its vertices, the C-PLANARITY problem asks whether there exist a planar drawing of G and a representation of each cluster as a topological disk enclosing all and only its vertices, such that no “unnecessary” crossings occur between disks and edges, or between disks. Ever since its introduction, this problem has been attracting a great deal of research. However, the question regarding its computational complexity withstood the attack of several powerful algorithmic tools, such as the Hanani-Tutte theorem [14, 16], the SPQR-tree machinery [10], and the Simultaneous PQ-ordering framework [6].

The clustering of a c-graph $\mathcal{C}(G, \mathcal{T})$ is described by a rooted tree \mathcal{T} whose leaves are the vertices of G and whose each internal node μ different from the root represents a *cluster* containing all and only the leaves of the subtree of \mathcal{T} rooted at μ . A c-graph is *flat* if \mathcal{T} has height 2. The *clusters-adjacency graph* G_A of a flat c-graph is the graph obtained from the c-graph by contracting each cluster into a single vertex, and by removing multi-edges and loops.

Cortese *et al.* [11] introduced a variant of C-PLANARITY for flat c-graphs, which we call C-PLANARITY WITH EMBEDDED PIPES. The input of this problem is a flat c-graph $\mathcal{C}(G, \mathcal{T})$ together with a planar drawing of its clusters-adjacency graph G_A , in which vertices of G_A are represented by disks and edges of G_A by pipes connecting the disks. The goal is then to produce a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$ in which each vertex of G lies inside the disk representing the cluster it belongs to and each inter-cluster edge of G is drawn inside the corresponding pipe. In [11] this problem is solved when the underlying graph G is a cycle. Chang, Erickson, and Xu [9] observed that in this case the problem is equivalent to determining whether a closed walk of length n in a simple plane graph is weakly simple, and improved the time complexity to $O(n \log n)$. The special case of the problem in which the clusters-adjacency graph is a path while G can be any planar graph, which is known by the name of STRIP PLANARITY, has also been studied. Polynomial-time algorithms for this problem have been presented when the underlying graph has a fixed planar embedding [2] and when it is a tree [14].

We remark that polynomial-time algorithms for the C-PLANARITY problem are known when strong limitations on the number or on the arrangement of the components of the clusters are imposed, where a *component* of a cluster $\mu \in \mathcal{T}$ is a maximal connected subgraph induced by the vertices of μ . In particular, C-PLANARITY can be decided in linear time when each cluster contains one component [10, 13] (the c-graph is *c-connected*). However, even when each cluster contains at most two components, polynomial-time algorithms are known only when further restrictions are imposed on the c-graph [6, 15]. The results we show in this paper are also based on imposing constraints on the number and combination of certain types of components.

A component of a cluster $\mu \in \mathcal{T}$ is *multi-edge* if it is incident to at least two inter-cluster edges, otherwise it is *single-edge*. Also, it is *passing* if it is adjacent to vertices belonging to at least two clusters in \mathcal{T} different from μ . Otherwise, it is adjacent to vertices of a unique cluster $\nu \in \mathcal{T}$ different from μ ; in this case, we say that it

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is *originating from* μ to ν . For STRIP PLANARITY the originating components can be further distinguished into *source* and *sink* components, based on whether ν corresponds to the strip above of below the one of μ .

Our contributions. We show that STRIP PLANARITY is polynomial-time solvable for instances with a unique source component (Section 3) and that C-PLANARITY WITH EMBEDDED PIPES is polynomial-time solvable for instances such that, for each cluster $\mu \in \mathcal{T}$ and for each edge (μ, ν) in G_A , either cluster μ contains at most one originating multi-edge component from μ to ν , or it contains at most two multi-edge originating components from μ to ν and does not contain any passing component that is incident to ν (Section 4). Finally, in Section 5 we introduce a generalization of C-PLANARITY WITH EMBEDDED PIPES, which we call C-PLANARITY WITH PIPES. Given a c-graph $\mathcal{C}(G, \mathcal{T})$, the goal of this problem is to find a planar drawing of the clusters-adjacency graph of $\mathcal{C}(G, \mathcal{T})$ whose vertices and edges are represented by disks and pipes, respectively, that allows for a drawing of $\mathcal{C}(G, \mathcal{T})$ that is a solution of C-PLANARITY WITH EMBEDDED PIPES. In other words, the goal is to find a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$ in which the inter-cluster edges are still required to be grouped into pipes, but the order of the pipes around each disk is not prescribed by the input. By introducing a new characterization of C-PLANARITY, we give an FPT algorithm for C-PLANARITY WITH PIPES that runs in $g(K, c) \cdot O(n^2)$ time, with $g(K, c) \in O(K^{c(K-2)})$, where K is the maximum number of multi-edge components in a cluster and c is the number of clusters with at least two multi-edge components. We remark that our results imply polynomial-time testing algorithms for all the three problems in the case in which each cluster contains at most two components.

2 Preliminaries

For the standard definitions on planar graphs, planar drawings, planar embeddings, and connectivity we point the reader to [12]. We use the term *rotation scheme* to denote the clockwise circular ordering of the edges incident to each vertex in a planar embedding, and refer to the containment relationships between vertices and cycles of the graph in the embedding as *relative positions*. Further, we say that a block of a 1-connected graph is *trivial* if it consists of a single edge, otherwise it is *non-trivial*.

PQ-trees. A PQ-tree T is an unrooted tree whose leaves are the elements of a ground set A . The internal nodes of T are either *P-nodes* or *Q-nodes*. PQ-tree T can be used to represent all and only the circular orderings $\mathcal{O}(T)$ on A satisfying a given set of *consecutivity constraints* on A , each of which specifies that a subset of the elements of A has to appear consecutively in all the sought circular orderings on A . The orderings in $\mathcal{O}(T)$ are all and only the circular orderings on the leaves of T obtained by arbitrarily ordering the neighbours of each P-node and by arbitrarily selecting for each Q-node a given circular ordering on its neighbours or its reverse ordering. PQ-trees were originally introduced by Booth and Lueker [8] in a rooted version.

Connectivity. A k -cut of a graph is a set of at most k vertices whose removal disconnects the graph. A connected graph is *biconnected* if it has no 1-cut. The maximal biconnected components of a graph are its *blocks*. Without loss of generality, in the following we assume that the clusters-adjacency graph G_A of $\mathcal{C}(G, \mathcal{T})$ is connected and that, for every cluster $\mu \in \mathcal{T}$ and for every component c of μ , it holds that: (i) there exists at least an inter-cluster edge incident to c , (ii) every block of c that is a leaf in the block-cut-vertex tree of c contains at least a vertex v such that v is not a cut-vertex of c and it is incident to at least an inter-cluster edge, and (iii) if there exists exactly one vertex in c that is incident to inter-cluster edges, then c consists of a single vertex.

Simultaneous Embedding with Fixed Edges. Given planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, the SEFE problem asks whether there exist planar drawings Γ_1 of G_1 and Γ_2 of G_2 such that (i) any vertex $v \in V$ is mapped to the same point in Γ_1 and Γ_2 and (ii) any edge $e \in E_1 \cap E_2$ is mapped to the same curve in Γ_1 and Γ_2 . We call *common graph* and *union graph* the graphs $G_\cap = (V, E_1 \cap E_2)$ and $G_\cup = (V, E_1 \cup E_2)$, respectively. See [5] for a survey.

We state a theorem on SEFE that will be fundamental for our results. Even though this theorem has never been explicitly stated in the literature, it can be easily deduced from known results [7], as discussed in the following.

Theorem 1. *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs whose common graph $G_\cap = (V, E_1 \cap E_2)$ is a forest and whose cut-vertices are incident to at most two non-trivial blocks. It can be tested in $O(|V|^2)$ time whether $\langle G_1, G_2 \rangle$ admits a SEFE.*

In particular, the theorem descends from a straightforward extension of the algorithm [7] to test SEFE of two biconnected planar graphs whose common graph is connected, to the case in which the common graph is a forest.

First, consider the following characterization of SEFE for two planar graphs.

Theorem 2 (Jünger and Schulz¹, Theorem 4). *Two planar graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ with common graph $G_\cap = (V, E_1 \cap E_2)$ have a SEFE if and only if they admit combinatorial embeddings inducing the same combinatorial embedding on G_\cap .*

¹M. Jünger and M. Schulz. Intersection graphs in simultaneous embedding with fixed edges. *J. Graph Algorithms Appl.*, 13(2):205218, 2009.

Recall that a combinatorial embedding of a planar graph $G(V, E)$ is defined by (i) the rotation scheme of each vertex in V and by (ii) the relative positions of the connected components of G . Hence, if G is acyclic, then a combinatorial embedding of G is entirely defined by (i). This fact and Theorem 2 imply the following.

Corollary 1. *Two planar graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ whose common graph $G_\cap = (V, E_1 \cap E_2)$ is a forest have a SEFE if and only if they admit combinatorial embeddings inducing the same rotation scheme on G_\cap .*

Bläsius and Rutter proved [7] that it can be tested in quadratic time whether two biconnected planar graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ admit combinatorial embeddings \mathcal{E}_1 and \mathcal{E}_2 , respectively, such that the rotation scheme of each vertex in V is the same in \mathcal{E}_1 and in \mathcal{E}_2 , when restricted to the common edges. They also proved that such a result extends to the case in which G_1 and G_2 have cut-vertices incident to at most two non-trivial blocks. Hence, Theorem 1 directly follows from Corollary 1 and from the results in [7].

3 Single-Source Strip Planarity

In this section we prove a result of the same flavour as that by Bertolazzi *et al.* [4] for the upward planarity testing of single-source digraphs. Namely, we show that instances of STRIP PLANARITY with a unique source component can be tested efficiently. The STRIP PLANARITY problem takes in input a pair $\langle G = (V, E), \gamma \rangle$, where $G = (V, E)$ is a planar graph and $\gamma : V \rightarrow \{1, \dots, k\}$ is a mapping of each vertex to one of k unbounded horizontal strips of the plane such that, for any two adjacent vertices $u, v \in V$, it holds that $|\gamma(u) - \gamma(v)| \leq 1$. The goal is to find a planar drawing of G in which vertices lie inside the corresponding strips and edges cross the boundary of any strip at most once. We observe that STRIP PLANARITY is equivalent to C-PLANARITY WITH EMBEDDED PIPES when G_A is a path [2].

We start with an auxiliary lemma. We say that an instance $\langle G, \gamma \rangle$ of STRIP PLANARITY on $k > 1$ strips is *spined* if there exists a path (v_1, \dots, v_k) in G such that $\gamma(v_i) = i$, vertex v_k is the unique vertex in the k -th strip, and each vertex v_i with $i \neq 1$ induces a component in the i -th strip; see Fig. 1. We call path (v_1, \dots, v_k) the *spine path* of $\langle G, \gamma \rangle$ and refer to edge (v_i, v_{i+1}) as the *i -th edge* of such a path.

Lemma 1. *Any positive spined instance $\langle G, \gamma \rangle$ of STRIP PLANARITY admits a strip-planar drawing in which the intersection point between the first edge of the spine path of $\langle G, \gamma \rangle$ and the horizontal line separating the first and the second strip is the left-most intersection point between any inter-strip edge and such a line.*

Proof. Let Γ be a strip-planar drawing of $\langle G, \gamma \rangle$; see Fig. 1(a). We show how to construct a strip-planar drawing Γ' of $\langle G, \gamma \rangle$ satisfying the requirements of the lemma. Consider two horizontal lines l' and l'' , with l' below l'' , that lie above any vertex in the first strip and below the horizontal line separating the first and the second strip in Γ , and that intersect any edge of G at most once. Denote by p_1, \dots, p_m (by q_1, \dots, q_m) the intersection points between l' (between l'') and the edges of G in the left-to-right order along l' (along l''). Let \mathcal{R}' (\mathcal{R}'') be the region delimited by l' (by l'') and by the horizontal line separating the second last and the last strip, and lying to the left of the spine path (v_1, \dots, v_k) of $\langle G, \gamma \rangle$ in Γ .

We obtain Γ' as follows; see Fig. 1(b). Initialize Γ' to Γ . Remove from Γ' the drawing of the part of G in the interior of \mathcal{R}' . Then, consider the drawing $\Gamma_{\mathcal{R}''}$ of G in the interior of \mathcal{R}'' in Γ . Add to Γ' a copy of drawing $\Gamma_{\mathcal{R}''}$ to the right of Γ , after a horizontal mirroring. Let q'_i be the point of Γ' corresponding to the mirrored and translated copy of point q_i . Finally, complete the drawing of the inter-strip edges crossing lines l' and l'' as curves between points p_i and q'_i in the interior of the first strip in Γ . The fact that such curves can be drawn in Γ' without introducing any crossings and without crossing the horizontal line separating the first and the second strip is due to the fact that points q'_1, \dots, q'_m appear in this right-to-left order along l'' and points p_1, \dots, p_m appear in this left-to-right order along l' . \square

Lemma 2. *Let $\langle G = (V, E), \gamma \rangle$ be a spined instance of STRIP PLANARITY on $k > 1$ strips with a unique source component c . It is possible to construct in linear time an equivalent spined instance $\langle G' = (V', E'), \gamma' \rangle$ of STRIP PLANARITY on $k - 1$ strips with a unique source component c' .*

Proof. Consider the source component c of $\langle G, \gamma \rangle$, which lies in the first strip. First, construct an auxiliary planar graph G_c as follows. Initialize $G_c = c$ and add a dummy vertex v to it. For each intra-strip edge e incident to a vertex u in c , add to G_c a dummy vertex v_e and edges (v, v_e) and (v_e, u) . If G_c contains cut-vertices, then let B_c be the block of G_c that contains v . Then, construct a PQ-tree \mathcal{T}_c representing all possible orders of the edges around v in a planar embedding of B_c . This can be done by applying the planarity testing algorithm of Booth and Lueker [8], in such a way that vertex v is the last vertex of the st -numbering of block B_c . Observe that each leaf of PQ-tree \mathcal{T}_c corresponds to exactly one vertex v_e in B_c . We construct a *representative graph* $G_{\mathcal{T}_c}$ from \mathcal{T}_c , as described in [13], composed of (i) *wheel graphs* (that is, graphs consisting of a cycle, called *rim*, and of a *central*

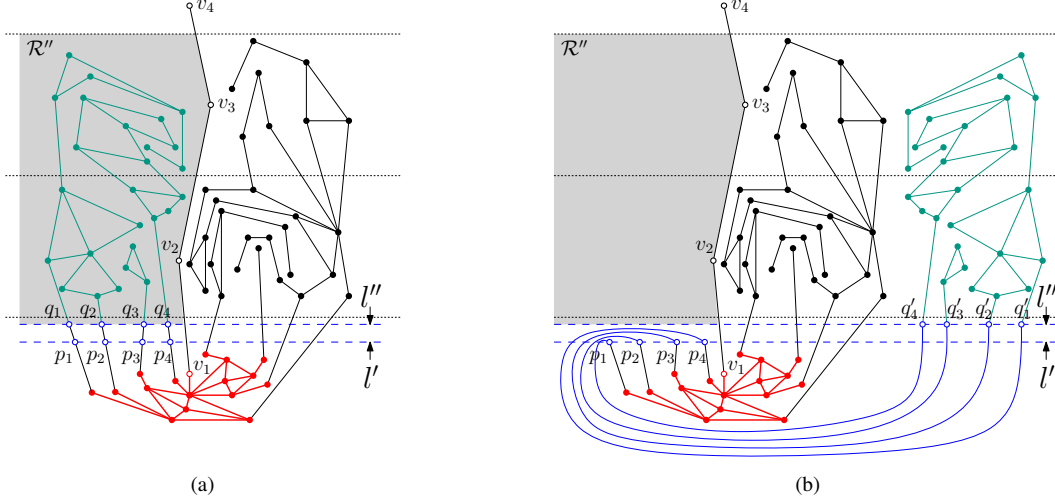


Figure 1: Illustrations for the proof of Lemma 3.

vertex connected to every vertex of the rim), of (ii) edges connecting vertices of different rims not creating any simple cycle that contains vertices belonging to more than one wheel, and of (iii) vertices of degree 1, which are in one-to-one correspondence with the leaves of \mathcal{T}_c (and hence with the dummy vertices v_e in B_c), each connected to a vertex of some rim. As proved in [13], in any planar embedding of $G_{\mathcal{T}_c}$ in which all the degree-1 vertices are incident to the same face, the order in which such vertices appear in a Eulerian tour of this face is in $O(\mathcal{T}_c)$.

We construct $\langle G', \gamma' \rangle$ as follows. For $i = 2, \dots, k$ and for each vertex v such that $\gamma(v) = i$, we add v to V' and we set $\gamma'(v) = i - 1$, that is, we assign all the vertices of the i -th strip of $\langle G, \gamma \rangle$, with $i \geq 2$, to the $(i - 1)$ -th strip of $\langle G', \gamma' \rangle$. Further, we add to E' all edges in $E \cap (V' \times V')$. Also, we add all the vertices and edges of $G_{\mathcal{T}_c}$ to V' and to E' , respectively, and we set $\gamma'(u) = 1$, for each vertex u of $G_{\mathcal{T}_c}$. Finally, for each inter-strip edge $e = (x, y)$ in E with $\gamma(x) = 1$ and $\gamma(y) = 2$, we add to E' an intra-strip edge between vertex y and the degree-1 vertex of $G_{\mathcal{T}_c}$ corresponding to v_e .

The construction of instance $\langle G', \gamma' \rangle$ can be carried out in linear time since the construction of \mathcal{T}_c takes linear time in the size of B_c [8] and since the construction of $G_{\mathcal{T}_c}$ takes linear time in the size of \mathcal{T}_c [13]. Hence, instance $\langle G', \gamma' \rangle$ has size linear in the size of $\langle G, \gamma \rangle$. Further, instance $\langle G', \gamma' \rangle$ has a unique source component, which contains $G_{\mathcal{T}_c}$ as a subgraph. This is due to the fact that any component in the second strip of $\langle G, \gamma \rangle$ has an inter-strip edge incident to a vertex of c . Finally, $\langle G', \gamma' \rangle$ is a spined instance whose spine path is the one obtained from the spine path of $\langle G, \gamma \rangle$ by removing its first edge.

We now show the equivalence between the two instances.

Suppose that $\langle G, \gamma \rangle$ admits a strip-planar drawing Γ , we show how to construct a strip-planar drawing Γ' of $\langle G', \gamma' \rangle$. First, observe that all the vertices of c incident to inter-strip edges lie on the outer face of the drawing of c in Γ . We subdivide each inter-strip edge incident to c with a dummy vertex v_e lying in the interior of the first strip of Γ . By the construction of \mathcal{T}_c and of $G_{\mathcal{T}_c}$, each degree-1 vertex of $G_{\mathcal{T}_c}$ corresponds to exactly one vertex v_e . Further, let c^+ be the subgraph of G induced by the vertices in c and by all the vertices v_e . Note that the order in which the vertices v_e appear in a Eulerian tour of the outer face of c^+ in Γ is in $O(\mathcal{T}_c)$. Hence, we can replace the drawing of c^+ in Γ with a drawing of $G_{\mathcal{T}_c}$ in which each degree-1 vertex is mapped to the vertex v_e it corresponds to. Finally, we obtain Γ' by merging the first two strips of Γ into the first strip of Γ' .

Suppose that $\langle G', \gamma' \rangle$ admits a strip-planar drawing Γ' ; we show how to construct a strip-planar drawing Γ of $\langle G, \gamma \rangle$. First, by Lemma 1, we can assume that in Γ' the intersection point between the first edge of the spine path of $\langle G', \gamma' \rangle$ and the line separating the first and the second strip in Γ' is the left-most intersection point between any edge (x, y) with $\gamma(x) = 1$ and $\gamma(y) = 2$ and such a line. Further, we can assume the following.

Claim 1. *The rim of every wheel W in $G_{\mathcal{T}_c}$ contains in its interior the central vertex of W and no other vertex in Γ' .*

Proof. The claim can be proved with the same techniques used in [3], by redrawing each edge connecting two adjacent vertices of the rim as a curve arbitrarily close to the length-2 path connecting them and passing through the central vertex of the wheel they belong to. This implies that all the degree-1 vertices of $G_{\mathcal{T}_c}$ lie in the outer face of the drawing of $G_{\mathcal{T}_c}$ induced by Γ' . \square

We obtain Γ as follows. We initialize Γ as the drawing in Γ' of the subinstance of $\langle G', \gamma' \rangle$ induced by the vertices not in $G_{\mathcal{T}_c}$, where the i -th strip in Γ' is mapped to the $(i+1)$ -th strip in Γ . First, we add a drawing of $G_{\mathcal{T}_c}$ in the first strip of Γ that is a copy of the drawing of $G_{\mathcal{T}_c}$ in Γ' . We now show how to draw in Γ the inter-strip edges incident to $G_{\mathcal{T}_c}$. Observe that these edges correspond to the intra-strip edge incident to $G_{\mathcal{T}_c}$ in Γ' . We draw each inter-strip edge (x, y) with y in $G_{\mathcal{T}_c}$ as a curve composed of six parts. The first part coincides with the drawing of (x, y) in Γ' ; the second part is a curve arbitrarily close to the drawing in Γ' of a path in $G_{\mathcal{T}_c}$ from y to the first vertex v_1 of the spine path of $\langle G', \gamma' \rangle$; the third part is a curve arbitrarily close to the drawing in Γ' of the first edge of the spine path of $\langle G', \gamma' \rangle$ till a point p in the interior of first strip of Γ' and arbitrarily close to the boundary of the second strip of Γ' ; the fourth part is a horizontal segment connecting p to a point q lying to the left of Γ' ; the fifth part is a vertical segment connecting q to a point r in the interior of the first strip of Γ ; and, finally, the sixth part is a curve connecting r to y . Observe that, by Claim 1, all the degree-1 vertices of $G_{\mathcal{T}_c}$ lie on its outer face in Γ' (and hence in Γ). Thus, it is possible to draw all the inter-strip edges incident to $G_{\mathcal{T}_c}$ without introducing any crossings, since the curves representing these edges preserve the same containment relationship between vertices and cycles in Γ as the corresponding intra-strip edges in Γ' .

To obtain a strip-planar drawing of $\langle G, \gamma \rangle$ we proceed as follows. Let H be the graph obtained from B_c by subdividing each edge e incident to v with a dummy vertex v_e and by removing v . We replace the drawing of $G_{\mathcal{T}_c}$ in Γ with a planar drawing of H such that the vertices v_e appear in a Eulerian tour of its outer face in the same clockwise order as the corresponding degree-1 vertices appear in a Eulerian tour of the outer face of $G_{\mathcal{T}_c}$ in Γ . Recall that these vertices are on the outer face of $G_{\mathcal{T}_c}$ in Γ , by Claim 1. Such a drawing of H exists since this order is in $O(\mathcal{T}_c)$ [13]. Finally, to complete Γ , for each cut-vertex z of G_c separating B_c from a subgraph G_z of G_c , we draw graph G_z arbitrarily close to z . This is possible since none of the vertices of G_z , except possibly for z , is incident to an inter-strip edge. This concludes the proof of the lemma. \square

Let $\langle G, \gamma \rangle$ be an instance of STRIP PLANARITY on $k > 1$ strips satisfying the properties of Lemma 2. By applying $k - 1$ times the transformation of this lemma, we obtain an instance of STRIP PLANARITY on $k = 1$ strips, that is, an instance whose strip-planarity coincides with the planarity of its underlying graph, which can be tested in linear time [8]. Hence, we get the following.

Lemma 3. *Let $\langle G = (V, E), \gamma \rangle$ be a spined instance of STRIP PLANARITY on $k > 1$ strips with a unique source component c . It is possible to decide in $O(k \times n)$ time whether $\langle G, \gamma \rangle$ admits a strip-planar drawing.*

Given an instance of STRIP PLANARITY, one can create $O(n)$ spined instances by attaching the spine path to each of the $O(n)$ vertices in the first strip. Hence, by Lemma 3, we get the following.

Theorem 3. *Let $\langle G, \gamma \rangle$ be an instance of STRIP PLANARITY on k strips such that there exists a unique source component c . It is possible to decide in $O(n^3)$ time whether $\langle G, \gamma \rangle$ admits a strip-planar drawing.*

Proof. Let v be a vertex of c . We define an instance $I_v = \langle G', \gamma' \rangle$ of STRIP PLANARITY on $k + 1$ strips as follows. For each vertex v of G , we add vertex v to $V(G')$ and set $\gamma'(v) = \gamma(v)$. Also, we add all the edges in $E(G)$ to $E(G')$. Finally, for $i = 2, \dots, k + 1$, we add to G' a vertex v_i and set $\gamma(v_i) = i$. Finally, we add edge (v, v_1) and edges (v_i, v_{i+1}) , for $i = 2, \dots, k$. Observe that, by construction, path (v, v_2, \dots, v_{k+1}) is such that v belongs to c and each vertex v_i , with $2 \leq i \leq k + 1$, induces a component in the i -th strip. Hence, I_v is a spined instance, which can thus be tested for strip-planarity in $O((k + 1) \cdot n)$ time, by Lemma 3.

It is not difficult to see that $\langle G, \gamma \rangle$ admits a strip-planar drawing if and only if there exists at least a vertex v of c that is incident to a vertex u with $\gamma(u) = 2$, such that instance I_v admits a strip-planar drawing. In fact, the *if* part follows from the fact that each I_v contains $\langle G, \gamma \rangle$ as a subinstance. The *only if* can be proved by observing that if $\langle G, \gamma \rangle$ admits a strip-planar drawing, then it also admits a strip-planar drawing Γ in which there exists a vertex v that is incident to a vertex u with $\gamma(u) = 2$ such that the intersection point between edge (u, v) and the line separating the first and the second strip in Γ is the left-most intersection point between any edge (x, y) with $\gamma(x) = 1$ and $\gamma(y) = 2$ and such a line in Γ .

The time bound descends from that of Lemma 3 and from the fact that (i) $k \in O(n)$ and $|I_v| \in O(|\langle G, \gamma \rangle|)$ and that (ii) since G is planar, the number of vertices in c that are incident to a vertex u with $\gamma(u) = 2$ is in $O(n)$. \square

4 C-Planarity with Embedded Pipes

In this section we show that the C-PLANARITY WITH EMBEDDED PIPES problem is solvable in quadratic time for a notable family of instances.

Let c be an originating component belonging to a cluster $\mu \in \mathcal{T}$ and let $\nu \neq \mu \in \mathcal{T}$ be the cluster to which the vertices of c are adjacent to. We say that c is *originating from μ to ν* .

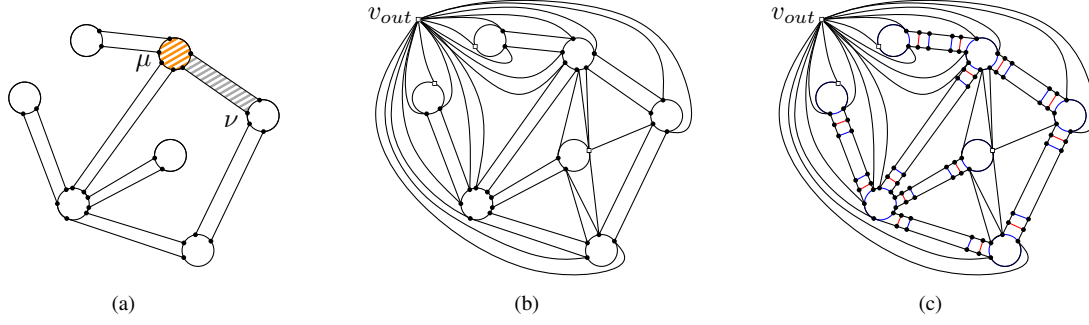


Figure 2: (a) Drawing Γ_A of the clusters-adjacency graph G_A of a flat c-graph; vertices have been placed at the intersection between clusters and pipes. The disk cycle for cluster μ and the pipe cycle for edge (μ, ν) of G_A are depicted as orange and grey tiled regions, respectively. (b) Frame gadget H . (c) Partial instance $\langle G_1, G_2 \rangle$ of SEFE constructed starting from Γ_A ; graphs G_1 , G_2 , and G_\cap are subdivisions of a triconnected planar graph.

Lemma 4. *Let $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ be an instance of C-PLANARITY WITH EMBEDDED PIPES and let \mathcal{S} be the maximum number of originating multi-edge components in a cluster that are incident to the same pipe. It is possible to construct in linear time an equivalent instance $\langle G_1, G_2 \rangle$ of SEFE such that (i) G_\cap is a spanning forest, (ii) each cut-vertex of $G_2 = (V, E_2)$ is incident to at most one non-trivial block, and (iii) each cut-vertex of $G_1 = (V, E_1)$ is incident to at most \mathcal{S} non-trivial blocks.*

Proof. We show how to construct $\langle G_1, G_2 \rangle$ starting from $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$. The frame gadget H is an embedded planar graph defined as follows. For each intersection point between a disk representing a cluster $\mu \in \mathcal{T}$ and a segment delimiting a pipe representing an edge of G_A incident to μ in the drawing Γ_A of G_A (see Fig. 2(a)), we add a vertex at this point. This results in a planar drawing of a graph; we set H to be this graph. We call *disk cycle* of μ the cycle in H obtained from the disk of μ in Γ_A . Similarly, we call *pipe cycle* of an edge (μ, ν) of G_A the cycle in H obtained from the pipe representing edge (μ, ν) in Γ_A . See Fig. 2(a). Observe that, for each cluster that is incident to exactly one pipe, this operation introduced two copies of the same edge; we subdivide with a dummy vertex the copy that is not incident to the interior of this pipe. Further, we add a vertex v_{out} in the outer face of H and connect it to all the vertices incident to this face. Finally, we triangulate all the faces of H that do not correspond to the interior of any cluster cycle or of any pipe cycle, hence obtaining a triconnected embedded planar graph. See Fig. 2(b).

We initialize $G_\cap = H$. For each edge $e \in E(H)$ separating the interior of a pipe from the interior of a disk, we remove e from G_1 (thus, edge e only belongs to G_2). Note that the definition of disk cycles and of pipe cycles can be extended to cycles in G_2 . Further, for each two edges e' and e'' corresponding to the two segments $(u_{\mu,\nu}, u_{\nu,\mu})$ and $(v_{\mu,\nu}, v_{\nu,\mu})$ delimiting a pipe representing an edge (μ, ν) of G_A , we subdivide e' with four dummy vertices $a'_{\mu,\nu}, b'_{\mu,\nu}, b'_{\nu,\mu}, a'_{\nu,\mu}$ and e'' with four dummy vertices $a''_{\mu,\nu}, b''_{\mu,\nu}, b''_{\nu,\mu}, a''_{\nu,\mu}$, and add edges $(a'_{\mu,\nu}, a''_{\mu,\nu})$ and $(a'_{\nu,\mu}, a''_{\nu,\mu})$ to G_1 and edges $(b'_{\mu,\nu}, b''_{\mu,\nu})$ and $(b'_{\nu,\mu}, b''_{\nu,\mu})$ to G_2 .

For each cluster $\mu \in \mathcal{T}$, we augment $\langle G_1, G_2 \rangle$ as follows; see Fig. 3. We subdivide an edge of G_\cap that corresponds to a portion of the boundary of the disk representing μ in Γ_A with a dummy vertex γ_μ , and we add to G_\cap a star C_μ , whose central vertex is adjacent to γ_μ , having a leaf $z(c_i)$ for each multi-edge component c_i of μ . Further, we add to G_\cap each component c_i of μ . Finally, for each edge (μ, ν) of G_A , we subdivide edge $(v_{\mu,\nu}, a'_{\mu,\nu})$ with a dummy vertex $\alpha_{\mu,\nu}$ and edge $(a''_{\mu,\nu}, b''_{\mu,\nu})$ with a dummy vertex $\beta_{\mu,\nu}$. Then, we add to G_\cap a star $A_{\mu,\nu}$ ($B_{\mu,\nu}$), whose central vertex is adjacent to vertex $\alpha_{\mu,\nu}$ (is identified with vertex $\beta_{\mu,\nu}$), with a leaf $a_\mu(e)$ (a leaf $b_\mu(e)$) for each inter-cluster edge e incident to a component of μ and to a component in ν . Also, $\langle G_1, G_2 \rangle$ contains the following edges only belonging to G_1 and to G_2 . For each inter-cluster edge $e = (x, y)$ with $x \in \mu$ and $y \in \nu$, we add to G_1 edges $(x, a_\mu(e))$, $(y, a_\nu(e))$, and $(b_\mu(e), b_\nu(e))$, and we add to G_2 edges $(a_\mu(e), b_\mu(e))$ and $(a_\nu(e), b_\nu(e))$. Also, for each vertex x of a component c_i of a cluster μ such that x is incident to at least an inter-cluster edge, we add to G_2 an edge $(x, z(c_i))$.

Clearly, $\langle G_1, G_2 \rangle$ can be constructed in linear time. We now prove that G_1 and G_2 satisfy the properties of the lemma. We note that G_1 and G_2 are connected, since each vertex of a component c_i is connected to the frame gadget by means of paths in G_1 and in G_2 passing through stars $A_{\mu,\nu}$ and C_μ , respectively. Also, for each cluster $\mu \in \mathcal{T}$, graph G_2 contains cut-vertices γ_μ , the center of star C_μ , and vertices $z(c_i)$, for each component c_i of μ . We now show that all these cut-vertices are incident to at most two non-trivial blocks of G_2 . Vertex γ_μ is incident to exactly one non-trivial block, that is, the one containing all the vertices and edges of the frame gadget. The center of C_μ is incident only to non-trivial blocks. Finally, vertices $z(c_i)$, for each component c_i of μ , are incident

of the components c_i of μ lie in the interior of such a cycle, by the connectivity of G_2 . 2. For each two clusters $\mu, \nu \in \mathcal{T}$, the vertices of the components of μ and of the components of ν lie in the interior of different cycles of G_1 . This is due to the fact that all the components of each cluster μ are connected by means of paths in G_1 to the leaves of a star $A_{\mu, \xi}$, where ξ is a cluster adjacent to μ . Also, all the leaves of these stars lie in the interior of a cycle of G_1 delimited by edges of G_\cap and by edges $(a'_{\mu, \xi_i}, a''_{\mu, \xi_i})$, for all the clusters ξ_i adjacent to μ . 3. For each inter-cluster edge e connecting a vertex v of a component c_i of cluster μ to a cluster ν , edge $(v, a_\mu(e))$ in G_1 crosses edge $(u_{\mu, \nu}, v_{\mu, \nu})$. This is due to the previous two points and the fact that the leaves of $A_{\mu, \nu}$ lie outside the disk cycle of μ . Note that we can assume that each of these edges crosses edge $(u_{\mu, \nu}, v_{\mu, \nu})$ exactly once, as otherwise we could redraw them in such a way to fulfil this requirement. 4. For two adjacent clusters $\mu, \nu \in \mathcal{T}$, the order in which the edges in G_1 incident to the leaves of $A_{\mu, \nu}$ cross edge $(u_{\mu, \nu}, v_{\mu, \nu})$ from $u_{\mu, \nu}$ to $v_{\mu, \nu}$ is the reverse of the order in which the edges in G_1 incident to the leaves of $A_{\nu, \mu}$ cross edge $(u_{\nu, \mu}, v_{\nu, \mu})$ from $u_{\nu, \mu}$ to $v_{\nu, \mu}$, where the identification between an edge incident to a leaf $a_\mu(e)$ of $A_{\mu, \nu}$ and an edge incident to a leaf $a_\nu(e)$ of $A_{\nu, \mu}$ is based on the inter-cluster edge e they correspond to. This is due to the fact that the order in which the edges in G_1 incident to the leaves of $A_{\mu, \nu}$ cross edge $(u_{\mu, \nu}, v_{\mu, \nu})$ is transmitted to the leaves of $B_{\mu, \nu}$ via edges in G_2 connecting the leaves of $A_{\mu, \nu}$ to the leaves of $B_{\mu, \nu}$, then it is transmitted to the leaves of $B_{\nu, \mu}$ via edges in G_1 connecting the leaves of $B_{\mu, \nu}$ to the leaves of $B_{\nu, \mu}$, and finally to the leaves of $A_{\nu, \mu}$ via edges in G_2 connecting the leaves of $B_{\nu, \mu}$ to the leaves of $A_{\nu, \mu}$. Note that, all the leaves of these stars lie in the interior of the pipe cycle corresponding to the edge (μ, ν) of G_A .

We describe the correspondence between the SEFE $\langle \Gamma_1, \Gamma_2 \rangle$ of $\langle G_1, G_2 \rangle$ and the c-planar drawing with embedded pipes Γ of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$. For each $\mu \in \mathcal{T}$, we draw region $R(\mu)$ as the simple closed region whose boundary coincides with the drawing in Γ_2 of the disk cycle of μ . Each component c_i of a cluster μ has the same drawing in Γ as c_i in $\langle \Gamma_1, \Gamma_2 \rangle$. For each inter-cluster edge $e = (x, y)$ with $x \in \mu$ and $y \in \nu$, the portion of e in the interior of $R(\mu)$ (of $R(\nu)$) coincides with the drawing of edge $(x, a_\mu(e))$ (of edge $(y, a_\nu(e))$) between x (between y) and the intersection point of this edge with edge $(u_{\mu, \nu}, v_{\mu, \nu})$ (with edge $(u_{\nu, \mu}, v_{\nu, \mu})$). To complete the drawing of all the inter-cluster edges between μ and ν in the interior of the pipe representing edge (μ, ν) in G_A , we connect the intersection points between the corresponding edges in G_1 and edges $(u_{\mu, \nu}, v_{\mu, \nu})$ and $(u_{\nu, \mu}, v_{\nu, \mu})$ by means of a set of non-intersecting curves. This is possible since the order in which the edges in G_1 incident to the leaves of $A_{\mu, \nu}$ cross edge $(u_{\mu, \nu}, v_{\mu, \nu})$ from $u_{\mu, \nu}$ to $v_{\mu, \nu}$ is the reverse of the order in which the edges in G_1 incident to the leaves of $A_{\nu, \mu}$ cross edge $(u_{\nu, \mu}, v_{\nu, \mu})$ from $u_{\nu, \mu}$ to $v_{\nu, \mu}$. Hence, Γ is a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$. The fact that Γ can be continuously deformed into a c-planar drawing with embedded pipes of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ is due to the fact that the paths in G_\cap corresponding to the segments delimiting the pipes incident to each cluster $\mu \in \mathcal{T}$ appear in $\langle \Gamma_1, \Gamma_2 \rangle$ in the same clockwise order as the corresponding pipes appear around the disk representing μ in Γ_A .

For the other direction, the goal is to construct a SEFE $\langle \Gamma_1, \Gamma_2 \rangle$ of $\langle G_1, G_2 \rangle$ that satisfies all the properties describe above starting from a c-planar drawing with pipes Γ of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$. For each cluster $\mu \in \mathcal{T}$, we draw the disk cycle of μ as the boundary of the disk of μ in Γ_A . Also, for each edge (μ, ν) of G_A , we draw the corresponding pipe cycle as the boundary of the pipe of edge (μ, ν) in Γ_A . For each cluster $\mu \in \mathcal{T}$, each component c_i of μ has the same drawing in $\langle \Gamma_1, \Gamma_2 \rangle$ as c_i in Γ . For each edge (μ, ν) of G_A , the stars $A_{\mu, \nu}$, $B_{\mu, \nu}$, $A_{\nu, \mu}$, and $B_{\nu, \mu}$ are drawn in $\langle \Gamma_1, \Gamma_2 \rangle$ in such a way that the order of their leaves is the same or the reverse of the order in which the inter-cluster edges between μ and ν traverse the boundary of the disk of μ in Γ . Note that this order is the reverse of the order in which these edges traverse the boundary of the disk of ν in Γ . This allows to draw all the edges in G_1 and in G_2 that are incident to such leaves without introducing any crossings between edges of the same graph. The drawing of star C_μ , for each cluster $\mu \in \mathcal{T}$, and of the edges in G_2 incident to its leaves can be easily obtained to respect the circular order of the inter-cluster edges incident to each of the components of μ . This concludes the proof of the lemma. \square

By Lemma 4 and Theorem 1 we have the following main result.

Theorem 4. C-PLANARITY WITH EMBEDDED PIPES can be solved in $O(n^2)$ time for instances $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ such that for each cluster $\mu \in \mathcal{T}$ and for each edge (μ, ν) in G_A either (CASE 1) cluster μ contains at most one originating multi-edge component from μ to ν or (CASE 2) cluster μ contains at most two multi-edge originating components from μ to ν and does not contain any passing component that is incident to ν .

Proof. Given an instance $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ of C-PLANARITY WITH EMBEDDED PIPES by Lemma 4 we can construct in linear time an equivalent instance $\langle G_1, G_2 \rangle$ of SEFE (whose size is hence linear in the size of $\mathcal{C}(G, \mathcal{T})$). Also, $\langle G_1, G_2 \rangle$ is such that G_\cap is a spanning forest, each cut-vertex of G_2 is incident to at most one non-trivial block, and each cut-vertex of G_1 is incident either to exactly one non-trivial block (CASE 1) or to at most two non-trivial blocks (CASE 2). Hence, we can apply Theorem 1 to decide in $O(|\mathcal{C}(G, \mathcal{T})|^2)$ time whether $\langle G_1, G_2 \rangle$ is a positive instance of SEFE (whether $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ is a positive instance of C-PLANARITY WITH EMBEDDED PIPES). \square

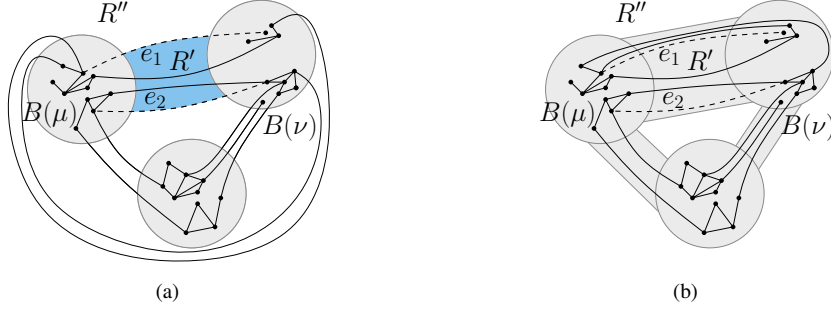


Figure 5: (a) A c-planar drawing with pipes Γ' . The two regions R' (blue) and R'' delimited by $B(\mu)$, by $B(\nu)$, and by edges e_1 and e_2 (dashed), where region R' does not contain any vertex of $G \setminus (\mu \cup \nu)$. (b) A c-planar drawing with pipes Γ^* corresponding to drawing Γ' in which inter-cluster edges are inside pipes.

5 C-Planarity with Pipes

In this section we introduce and study the C-PLANARITY WITH PIPES problem. A c-planar drawing Γ of a flat c-graph $\mathcal{C}(G, \mathcal{T})$ is a *c-planar drawing with pipes* of $\mathcal{C}(G, \mathcal{T})$ if, for any two clusters $\mu, \nu \in \mathcal{T}$ that are adjacent in G_A and for any two inter-cluster edges e_1 and e_2 that are incident to both μ and ν , one of the two regions delimited by $B(\mu)$, by $B(\nu)$, by e_1 , and by e_2 does not contain any vertex of $G \setminus (\mu \cup \nu)$; two examples are given in Figs. 5(a) and 5(b). The C-PLANARITY WITH PIPES problem asks for the existence of a c-planar drawing with pipes of a given flat c-graph.

Note that, if a c-graph $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing with pipes, then it is always possible to construct a drawing Γ_A of its clusters-adjacency graph G_A in which vertices and edges are represented by disks and pipes, respectively, such that $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ is a positive instance of C-PLANARITY WITH EMBEDDED PIPES; Fig. 5(b) shows a solution for the instance of C-PLANARITY WITH EMBEDDED PIPES determined by the c-planar drawing with pipes in Fig. 5(a). The following lemma proves that, by suitably augmenting the original c-graph $\mathcal{C}(G, \mathcal{T})$, it is possible to enforce that the resulting drawing Γ_A of G_A respects a specific embedding (if any solution determining a drawing respecting this embedding exists), which implies that C-PLANARITY WITH PIPES is in fact a generalization of C-PLANARITY WITH EMBEDDED PIPES.

Lemma 5. C-PLANARITY WITH EMBEDDED PIPES *reduces in linear time to C-PLANARITY WITH PIPES. The reduction does not increase the number of multi-edge components in any cluster.*

Proof. Let $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ be an instance of C-PLANARITY WITH EMBEDDED PIPES, where $\mathcal{C}(G, \mathcal{T})$ is a c-graph and Γ_A is planar drawing of G_A . We construct an equivalent instance $\mathcal{C}^*(G^*, \mathcal{T}^*)$ of C-PLANARITY WITH PIPES.

First, we initialize $\mathcal{C}(G, \mathcal{T}) = \mathcal{C}^*(G^*, \mathcal{T}^*)$. Then, we augment $\mathcal{C}^*(G^*, \mathcal{T}^*)$ by adding a matching to G^* in such a way that the clusters-adjacency graph G_A^* of $\mathcal{C}^*(G^*, \mathcal{T}^*)$ is a triangulated planar graph. In order to do so, we consider a triangulated planar graph G'_A obtained from G_A by adding edges in such a way that the restriction of the unique combinatorial embedding of G'_A to the edges of G_A is the same as the combinatorial embedding of G_A in Γ_A . For each new edge $e = (\mu, \nu)$ of $G'_A \setminus G_A$, we add to $\mathcal{C}^*(G^*, \mathcal{T}^*)$ a new vertex $\mu(e)$ to μ and a new vertex $\nu(e)$ to ν , and an inter-cluster edge $(\mu(e), \nu(e))$.

Clearly, the reduction can be performed in linear time and G_A^* coincides with G'_A . Also, vertices $\mu(e)$ and $\nu(e)$ are single-edge components of μ and ν , respectively, and thus the number of multi-edge components remains the same. Further, since G_A^* is triconnected, any c-planar drawing with pipes of $\mathcal{C}^*(G^*, \mathcal{T}^*)$ contains a c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$ in which the pipes appear in the desired order around each cluster.

Finally, it is not difficult to see that any c-planar drawing with pipes Γ of $\mathcal{C}(G, \mathcal{T})$ in which the order of the pipes incident to each cluster is the same as in Γ_A can be extended to a c-planar drawing with pipes of $\mathcal{C}^*(G^*, \mathcal{T}^*)$ by drawing the edges in $G'_A \setminus G_A$. In fact, for each of these edges (μ, ν) there exists a region of Γ delimited by a portion of $B(\mu)$ and a portion of $B(\nu)$ where this edge can be drawn, since there exists a face of Γ_A to which both μ and ν are incident. \square

In the remainder of the section we present an FPT algorithm for C-PLANARITY WITH PIPES in two parameters, namely the maximum number K of multi-edge components in a cluster and the number c of clusters with at least two multi-edge components. Our result is based on a characterization of C-PLANARITY of flat c-graphs in terms of a newly defined constrained embedding problem.

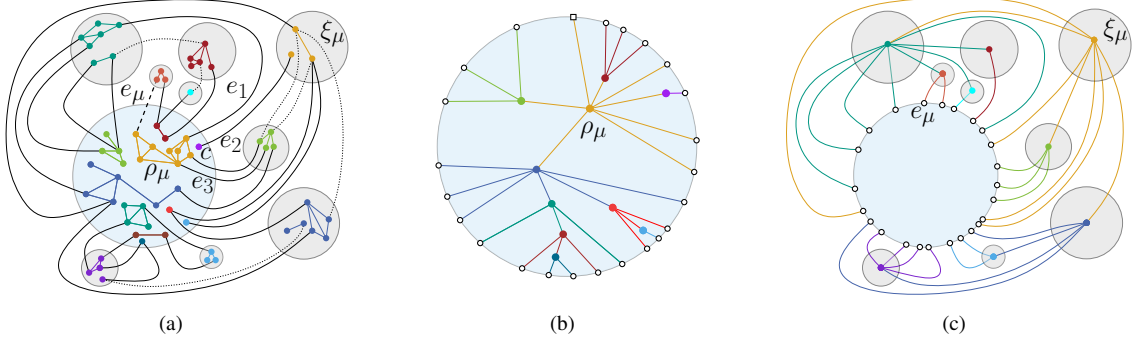


Figure 6: (a) A c-planar drawing Γ focused on cluster μ . Edges incident to μ are solid. Component c is nested into component ρ_μ . Trees (b) X_μ and (c) Y_μ such that Γ is consistent with X_μ and Y_μ .

5.1 A Characterization of Flat C-Planarity

We start with some definitions. Let $\mathcal{C}(G, \mathcal{T})$ be a flat c-graph and let μ be a cluster in \mathcal{T} . A *components tree* X_μ of μ is a rooted tree in which every internal vertex is a multi-edge component c of μ and in which every leaf $x_\mu(e)$ corresponds to an inter-cluster edge e incident to one of such components. A *neighbor-clusters tree* Y_μ of μ is a rooted tree in which there exists an internal vertex ν for each cluster ν adjacent to μ , plus a set of additional internal vertices, and in which every leaf $y_\mu(e)$ corresponds to an inter-cluster edge e incident to μ . Let Γ be a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$, let X_μ be a components tree of μ rooted at a multi-edge component ρ_μ , and let Y_μ be a neighbor-clusters tree of μ rooted at a cluster ξ_μ , such that there exists an inter-cluster edge e_μ incident to both ρ_μ and ξ_μ . Let \mathcal{O}_μ be the clockwise linear order in which the edges incident to μ traverse $B(\mu)$ in Γ , starting from and ending at e_μ . Drawing Γ is *consistent with* X_μ if, for each vertex $c \in X_\mu$, the leaves of the subtree of X_μ rooted at c are consecutive in the restriction of \mathcal{O}_μ to the inter-cluster edges incident to multi-edge components of μ . Also, Γ is *consistent with* Y_μ if, for each vertex $\nu \in Y_\mu$, the leaves of the subtree of Y_μ rooted at ν are consecutive in \mathcal{O}_μ . Let \mathcal{X} and \mathcal{Y} be two sets containing a components tree X_μ and a neighbor-clusters tree Y_μ , respectively, for each μ in \mathcal{T} . Drawing Γ is *consistent with* $\langle \mathcal{X}, \mathcal{Y} \rangle$ if, for each $\mu \in \mathcal{T}$, drawing Γ is consistent with both X_μ and Y_μ .

Given a flat c-graph $\mathcal{C}(G, \mathcal{T})$, together with two sets \mathcal{X} and \mathcal{Y} of components trees and of neighbor-clusters trees, respectively, for all the clusters in \mathcal{T} , problem INCLUSION-CONSTRAINED C-PLANARITY asks whether a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$ exists that is consistent with $\langle \mathcal{X}, \mathcal{Y} \rangle$.

Theorem 5. A flat c-graph $\mathcal{C}(G, \mathcal{T})$ is c-planar if and only if there exist two sets \mathcal{X} and \mathcal{Y} of components trees and of neighbor-clusters trees, respectively, for all the clusters in \mathcal{T} , such that $\langle \mathcal{C}(G, \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ is a positive instance of INCLUSION-CONSTRAINED C-PLANARITY.

Proof. One direction is trivial. Namely, if $\langle \mathcal{C}(G, \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ is a positive instance of INCLUSION-CONSTRAINED C-PLANARITY, then $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing (even one that is consistent with $\langle \mathcal{X}, \mathcal{Y} \rangle$).

We prove the other direction. Let Γ be a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$. Consider each cluster $\mu \in \mathcal{T}$. Suppose that there exists at least a multi-edge component ρ_μ in μ , as otherwise X_μ is the empty tree and Γ is trivially consistent with it. Let e_μ be any inter-cluster edge incident to ρ_μ . Let \mathcal{O}_μ be the clockwise linear order of the edges incident to μ starting from e_μ and ending at e_μ . Also, let ξ_μ be the cluster different from μ to which e_μ is incident. Since Γ is c-planar, there exist no four edges e_1, e_2, e_3 , and e_4 appearing in this order in \mathcal{O}_μ such that e_1 and e_3 are incident to a component c' of μ , and e_2 and e_4 are incident to a component $c'' \neq c'$ of μ . Hence, for each two components c' and c'' in μ , order \mathcal{O}_μ defines a unique “inclusion” hierarchy with respect to ρ_μ . Namely, we say that c' is *nested into* c'' if there exists three edges e_1, e_2 , and e_3 appearing in this order in \mathcal{O}_μ such that e_1 and e_3 are incident to c'' , and e_2 is incident to c' . Refer to Fig. 6(a).

Note that such a hierarchy is acyclic and that every component different from ρ_μ is nested into ρ_μ , since \mathcal{O}_μ start and ends at e_μ . We construct a tree X_μ rooted at ρ_μ in which every internal vertex is a multi-edge component c of μ and in which every leaf $x_\mu(e)$ corresponds to an inter-cluster edge e incident to one of such components; refer to Figs. 6(a) and 6(b). There exists an edge $(x_\mu(e), c)$ if and only if edge e is incident to a vertex of component $c \in X_\mu$. Also, there exists an edge (c', c'') if component $c' \in X_\mu$ is nested into component $c'' \in X_\mu$ and there exists no other component $c^* \in X_\mu$ such that c^* is nested into c'' and c' is nested into c^* in Γ . By construction, X_μ is a components tree and Γ is consistent with X_μ .

Similarly, order \mathcal{O}_μ determines whether any two clusters adjacent to μ in G_A are *nested* one into the other; this determines an acyclic hierarchy in which every cluster different from ξ_μ is nested into ξ_μ . We construct a tree Y_μ rooted at ξ_μ in which there exists an internal vertex ν for each cluster ν adjacent to μ in G_A and in which every

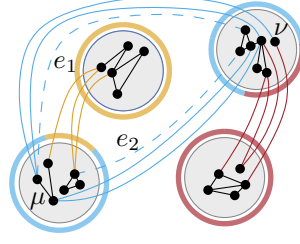


Figure 7: A c-planar drawing that is not a c-planar drawing with pipes, even if the inter-cluster edges incident to the same cluster are consecutive (see the annuli around clusters), due to the presence of trivial block (μ, ν) .

leaf $y_\mu(e)$ corresponds to an inter-cluster edge e that is incident to μ ; refer to Figs. 6(a) and 6(c). There exists an edge $(y_\mu(e), \nu)$ if and only if edge e is incident to a vertex of cluster $\nu \in Y_\mu$. Also, there exists an edge (ν', ν'') if cluster $\nu' \in Y_\mu$ is nested into cluster $\nu'' \in Y_\mu$ and there exists no other cluster ν^* such that ν^* is nested into ν'' and ν' is nested into ν^* in Γ . By construction, Y_μ is a neighbor-clusters tree and Γ is consistent with Y_μ . \square

In the following theorem, whose proof is deferred to Section 6, we show that the INCLUSION-CONSTRAINED C-PLANARITY problem can be solved efficiently.

Theorem 6. INCLUSION-CONSTRAINED C-PLANARITY can be solved in quadratic time.

In the following section we prove that, for each cluster μ of a c-graph $\mathcal{C}(G, \mathcal{T})$, there exists a unique neighbor-clusters tree Y_μ such that every c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$ is consistent with Y_μ . Hence, an FPT algorithm for C-PLANARITY WITH PIPES can be based on generating, for each cluster, all the possible components trees and its unique neighbor-clusters tree, and on testing the corresponding instances of INCLUSION-CONSTRAINED C-PLANARITY by Theorem 6.

5.2 Neighbor-clusters Trees in C-Planar Drawings with Pipes

In the following theorem we give a characterization of the c-graphs that are positive instances of C-PLANARITY WITH PIPES based on the possible orders of inter-cluster edges around each cluster in any c-planar drawing. We first consider only c-graphs whose clusters-adjacency graph G_A has no trivial blocks; however, we prove later that this is not a restriction.

Theorem 7. Let $\mathcal{C}(G, \mathcal{T})$ be a flat c-graph such that G_A has no trivial block. Then, $\mathcal{C}(G, \mathcal{T})$ is a positive instance of C-PLANARITY WITH PIPES if and only if $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing Γ in which, for each cluster $\mu \in \mathcal{T}$, the inter-cluster edges between μ and any cluster ν adjacent to μ in G_A are consecutive in the order in which the inter-cluster edges incident to μ cross $B(\mu)$ in Γ .

Proof. One direction is trivial, since any c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$ is a c-planar drawing satisfying the conditions of the theorem.

Suppose that $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing Γ satisfying the conditions of the theorem. We prove that Γ is a c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$. Assume for a contradiction that this is not the case, that is, there exist two clusters $\mu, \nu \in \mathcal{T}$ that are adjacent in G_A and two inter-cluster edges e_1 and e_2 that are incident to both μ and ν , such that both the regions delimited by $B(\mu)$, by $B(\nu)$, by e_1 , and by e_2 in Γ contain at least a vertex of $G \setminus (\mu \cup \nu)$; see Fig. 7.

Note that, if there exists a cluster that is adjacent to μ (to ν) in G_A in the interior of one of the two regions, then there exists no other cluster that is adjacent to μ (to ν) in G_A in the interior of the other region, as otherwise the edges between μ and ν would not be consecutive around $B(\mu)$ (around $B(\nu)$). Hence, for every cluster lying in the interior of one of the regions, all the paths in G_A connecting it to μ pass through ν ; also, for every cluster lying in the interior of the other region, all the paths in G_A connecting it to ν pass through μ . Therefore, (μ, ν) is a trivial block of G_A , a contradiction. \square

We exploit Theorem 7 to construct a neighbor-clusters tree Y_μ° of each cluster $\mu \in \mathcal{T}$ such that any c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$ is consistent with Y_μ° . Tree Y_μ° is rooted at a vertex ω_μ . There exists a child ν of ω_μ for each cluster ν adjacent to μ , having a leaf $y_\mu(e)$ for each inter-cluster edge e incident to μ and to ν . We call Y_μ° the *pipe-neighbor-clusters tree* of μ . Theorem 7 and the construction of Y_μ° , for each cluster $\mu \in \mathcal{T}$, imply the following.

Corollary 2. *Let $\mathcal{C}(G, \mathcal{T})$ be a c-graph whose clusters-adjacency graph has no trivial blocks. Then, $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing with pipes if and only if $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing Γ in which, for each $\mu \in \mathcal{T}$, drawing Γ is consistent with Y_μ° .*

Corollary 2 allows us to reduce the problem of testing C-PLANARITY WITH PIPES for a c-graph whose clusters-adjacency graph G_A has no trivial blocks to that of testing INCLUSION-CONSTRAINED C-PLANARITY, where the role played by the neighbor-clusters trees is now taken by the pipe-neighbor-clusters trees. Next, we overcome the requirement that G_A has no trivial block.

Lemma 6. *Let $\mathcal{C}(G, \mathcal{T})$ be an instance of C-PLANARITY WITH PIPES in which G_A contains trivial blocks. It is possible to construct in linear time an equivalent instance $\mathcal{C}^*(G^*, \mathcal{T}^*)$ of C-PLANARITY WITH PIPES in which G_A^* has no trivial block. Further, $K_* = K$ and $c_* = c$, where K (K_*) is the maximum number of multi-edge components in a cluster of $\mathcal{C}(G, \mathcal{T})$ (of $\mathcal{C}^*(G^*, \mathcal{T}^*)$) and c (c_*) is the number of clusters of $\mathcal{C}(G, \mathcal{T})$ (of $\mathcal{C}^*(G^*, \mathcal{T}^*)$) with at least two multi-edge components.*

Proof. Consider any trivial block (μ, ν) in G_A . We show how to construct an instance $\mathcal{C}^+(G^+, \mathcal{T}^+)$ of C-PLANARITY WITH PIPES equivalent to $\mathcal{C}(G, \mathcal{T})$ such that (i) the block of G_A^+ containing (μ, ν) is not a trivial block, (ii) G_A^+ does not contain any trivial block that does not belong to G_A , and (iii) $K_+ = K$ and $c_+ = c$, where K_+ is the maximum number of multi-edge components in a cluster of $\mathcal{C}^+(G^+, \mathcal{T}^+)$ and c_+ is the number of clusters of $\mathcal{C}^+(G^+, \mathcal{T}^+)$ with at least two multi-edge components. Repeating such a transformation eventually yields an instance $\mathcal{C}^*(G^*, \mathcal{T}^*)$ satisfying the required properties.

We initialize $\mathcal{C}^+(G^+, \mathcal{T}^+) = \mathcal{C}(G, \mathcal{T})$. Then, we add a new cluster η to \mathcal{T}^+ , which only contains a new vertex v . Also, we add a vertex u_μ to cluster μ and a vertex u_ν to cluster ν , and edges (v, u_μ) and (v, u_ν) to $\mathcal{C}^+(G^+, \mathcal{T}^+)$.

We prove that $\mathcal{C}^+(G^+, \mathcal{T}^+)$ and $\mathcal{C}(G, \mathcal{T})$ are equivalent. One direction is trivial, as any c-planar drawing with pipes of $\mathcal{C}^+(G^+, \mathcal{T}^+)$ contains a c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$.

Suppose that $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing with pipes Γ . Consider the two inter-cluster edges e_1 and e_2 adjacent to both μ and ν such that the region R_μ delimited by $B(\mu)$, by $B(\nu)$, by e_1 , and by e_2 containing all the vertices of $G \setminus (\mu \cup \nu)$ does not contain any other inter-cluster edge adjacent to both μ and ν in Γ . We construct a c-planar drawing with pipes Γ^+ of $\mathcal{C}^+(G^+, \mathcal{T}^+)$ starting from Γ . Namely, draw path (u_μ, v, u_ν) as a curve arbitrarily close to edge e_1 in Γ in the interior of region R_μ introducing neither edge-edge nor edge-region crossings, and draw $R(\eta)$ as a simple closed region enclosing only the vertex v .

The time bound descends from the fact that each augmentation step described above can be performed in constant time and that the number of trivial blocks in G_A is at most linear in the size of $\mathcal{C}(G, \mathcal{T})$.

Finally, $K_+ = K$ and $c_+ = c$, since u_μ and u_ν are single-edge components of μ and of ν , respectively, while η contains exactly one component, which is a multi-edge component. \square

5.3 An FPT Algorithm for C-Planarity with Pipes

In the following we prove the main result of the section.

Theorem 8. *C-PLANARITY WITH PIPES can be tested in $O(K^{c(K-2)}) \cdot O(n^2)$ time, where K is the maximum number of multi-edge components in a cluster and c is the number of clusters with at least two multi-edge components.*

Proof. Let $\mathcal{C}(G, \mathcal{T})$ be an instance of C-PLANARITY WITH PIPES. First, apply Lemma 6 to construct in linear time an equivalent instance $\mathcal{C}^*(G^*, \mathcal{T}^*)$ of C-PLANARITY WITH PIPES whose clusters-adjacency graph contains no trivial blocks (possibly $\mathcal{C}^*(G^*, \mathcal{T}^*) = \mathcal{C}(G, \mathcal{T})$) and such that $K_* = K$ and $c_* = c$, where K_* is the maximum number of multi-edge components in a cluster of $\mathcal{C}^*(G^*, \mathcal{T}^*)$ and c_* is the number of clusters of $\mathcal{C}^*(G^*, \mathcal{T}^*)$ with at least two multi-edge components. Second, construct the set \mathcal{Y} containing the unique pipe-neighbor-clusters tree Y_μ° , for each cluster $\mu \in \mathcal{T}^*$. Then, construct all the possible sets \mathcal{X} of components trees, for each cluster $\mu \in \mathcal{T}^*$, as follows. If μ does not contain any multi-edge component, then this set contains only the empty tree, while if μ contains exactly one multi-edge component c , then this set contains only a star whose central vertex is c , with a leaf $x_\mu(e)$ for each inter-cluster edge e incident to c . Otherwise, consider a set \mathcal{I} containing a vertex c for each multi-edge component c of μ . We generate all the trees on the vertices in \mathcal{I} and, for each of them, we add to each vertex c a leaf $x_\mu(e)$ for each inter-cluster edge e incident to c ; by Cayley's formula [1], the number of these trees is $k_\mu^{k_\mu-2}$, where k_μ is the number of multi-edge components of μ . Finally, apply Theorem 6 to test whether $\langle \mathcal{C}^*(G^*, \mathcal{T}^*), \mathcal{X}, \mathcal{Y} \rangle$ is a positive instance of INCLUSION-CONSTRAINED C-PLANARITY, for each pair $\langle \mathcal{X}, \mathcal{Y} \rangle$. By Theorem 7 and Corollary 2, we conclude that $\mathcal{C}^*(G^*, \mathcal{T}^*)$ is a positive instance of C-PLANARITY WITH PIPES if and only if at least one of such tests succeeds.

There exist $\prod_{\mu \in \mathcal{S}} k_\mu^{k_\mu - 2}$ combinations of components trees over all clusters in \mathcal{T} , where \mathcal{S} is the set of clusters in \mathcal{T} containing at least two multi-edge components, which we can upper bound by $K^{c(K-2)}$, where K is the maximum number of multi-edge components in a cluster and $c = |\mathcal{S}|$. Since there exists a unique set \mathcal{Y} of pipe-neighbor-clusters trees for $\mathcal{C}(G, \mathcal{T})$ and since each application of Theorem 6 requires quadratic time, the statement follows. \square

We observe two notable corollaries of Theorem 8 (for the second, see Lemma 5).

Corollary 3. STRIP PLANARITY can be tested in $O(K^{s(K-2)}) \cdot O(n^2)$ time, where K is the maximum number of multi-edge components in a strip and s is the number of strips containing at least two multi-edge components.

Corollary 4. C-PLANARITY WITH EMBEDDED PIPES can be tested in $K^{c(K-2)} \cdot O(n^2)$ time, where K is the maximum number of multi-edge components in a cluster and c is the number of clusters with at least two multi-edge components.

6 Proof of Theorem 6

In this section we give a proof of Theorem 6, which has been stated in Section 5.1, by describing an algorithm that is based on a linear-time reduction (Lemma 7) from instances of INCLUSION-CONSTRAINED C-PLANARITY to equivalent instances of SEFE that can be solved in quadratic time by Theorem 1. We first describe the reduction in Lemma 7 and then discuss its implications to complete the proof of Theorem 6.

Lemma 7. Let $\langle \mathcal{C}(G, \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ be an instance of INCLUSION-CONSTRAINED C-PLANARITY. It is possible to construct in linear time an equivalent instance $\langle G_1(V, E_1), G_2(V, E_2) \rangle$ of SEFE in which the common graph $G_\cap = (V, E_1 \cap E_2)$ is a forest such that the cut-vertices of G_1 and G_2 are incident to at most two non-trivial blocks.

Proof. For each cluster $\mu \in \mathcal{T}$ instance $\langle G_1, G_2 \rangle$ contains a *cluster gadget* G_μ composed of edges in $E_1 \cup E_2$. These gadgets are then attached by means of edges in E_2 to an outer *frame*, composed of edges of G_\cap , which enforces them to lie “outside of each other”. Finally, these gadgets are connected with each other by means of edges in E_1 representing inter-cluster edges.

Our reduction is inspired by the original reduction from C-PLANARITY to SEFE proposed by Schaefer [16]. However, while that reduction produces instances of SEFE in which the cut-vertices of G_1 and G_2 may have a linear number of non-trivial blocks, we exploit the presence of the components-tree and of the neighbor-clusters-tree to create instances in which the non-trivial blocks incident to cut-vertices are at most two, which makes instance $\langle G_1, G_2 \rangle$ polynomial-time solvable. We now describe in detail the construction of G_1 and G_2 .

For each cluster $\mu \in \mathcal{T}$, cluster gadget G_μ is constructed as follows. Refer to Fig. 8.

We first describe the part of G_μ that belongs to both G_1 and G_2 . Gadget G_μ contains a wheel W_μ with a *central vertex* c_μ that is connected to all the vertices of a cycle $(\rho_\mu^1, \rho_\mu^2, \delta_\mu, \rho_\mu^3, \alpha_\mu, \beta_\mu, \rho_\mu^4, \gamma_\mu, \rho_\mu^5, \delta'_\mu, \rho_\mu^6, \alpha'_\mu, \beta'_\mu, \rho_\mu^7)$, which is the *rim* of W_μ . Also, it contains a star A_μ (A'_μ), centered at α_μ (at α'_μ), with a leaf $a_\mu(e)$ (a leaf $a'_\mu(e)$) for each inter-cluster edge e incident to μ that is incident to a multi-edge component of μ . Then, G_μ contains a star B_μ (B'_μ), whose central vertex is adjacent to vertex β_μ (to vertex β'_μ), with a leaf $b_\mu(e)$ (a leaf $b'_\mu(e)$) for each inter-cluster edge e incident to a multi-edge component of μ . Further, it contains a star C_μ , whose central vertex is adjacent to vertex γ_μ , with a leaf $z(c_i)$ for each multi-edge component c_i of μ . Additionally, G_μ contains a copy of each multi-edge component c_i of μ . Gadget G_μ also contains trees $X_\mu \in \mathcal{X}$ and $Y_\mu \in \mathcal{Y}$; recall that, X_μ has a leaf $x_\mu(e)$ for each inter-cluster edge e incident to a multi-edge component of μ , while Y_μ has a leaf $y_\mu(e)$ for each inter-cluster edge e incident to μ . Finally, G_μ contains an edge $(y_\mu(e_\mu), \delta_\mu)$ and an edge $(x_\mu(e_\mu), \delta'_\mu)$, where e_μ is an arbitrary inter-cluster edge incident to the root ρ_μ of X_μ , if X_μ is not the empty tree, or an arbitrary inter-cluster edge incident to μ , otherwise.

We now describe the edges of G_μ only belonging to E_1 . Namely, E_1 contains an edge (ρ_μ^3, ρ_μ^6) . Also, for each inter-cluster edge e incident to a vertex v belonging to a multi-edge component of μ , set E_1 contains an edge $(v, x_\mu(e))$, an edge $(b_\mu(e), a_\mu(e))$, and an edge $(a'_\mu(e), b'_\mu(e))$.

Finally, we describe the edges of G_μ only belonging to E_2 . Namely, E_2 contains edges (ρ_μ^2, ρ_μ^7) and (ρ_μ^4, ρ_μ^5) . Also, for each vertex v of a multi-edge component c_i of μ such that v is incident to at least an inter-cluster edge, set E_2 contains an edge $(z(c_i), v)$. Further, for each inter-cluster edge e incident to a multi-edge component of μ , set E_2 contains an edge $(x_\mu(e), b_\mu(e))$, an edge $(a_\mu(e), a'_\mu(e))$, and an edge $(b'_\mu(e), y_\mu(e))$. Finally, for each inter-cluster edge e incident to a single-edge component of μ , set E_2 contains an edge connecting $y_\mu(e)$ with the center of star B'_μ . This concludes the construction of G_μ .

We then add to G_\cap a *frame* consisting of cycle $C = (\sigma_{\mu_1}, \dots, \sigma_{\mu_k}, \sigma^*)$, with $\mu_i \in \mathcal{T}$. Also, we add to E_1 an edge $(\sigma^*, \rho_{\mu_1}^1)$. Finally, we add to E_2 an edge $(\rho_{\mu_i}^1, \sigma_{\mu_i})$ for each cluster $\mu_i \in \mathcal{T}$. Refer to Fig. 9.

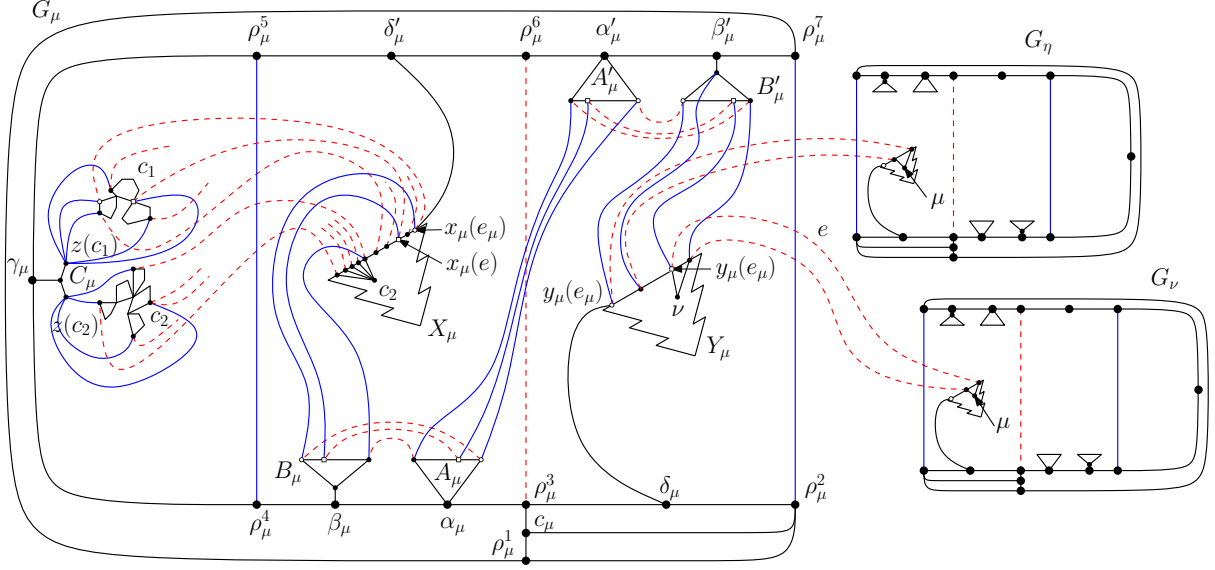


Figure 8: Sketch of the cluster gadgets G_μ , G_ν , and G_η for cluster μ and its neighbors ν and η . For readability purposes, edges of G_\cap in G_μ between the center c_μ of the wheel W_μ and some vertices of the rim of W_μ have been omitted.

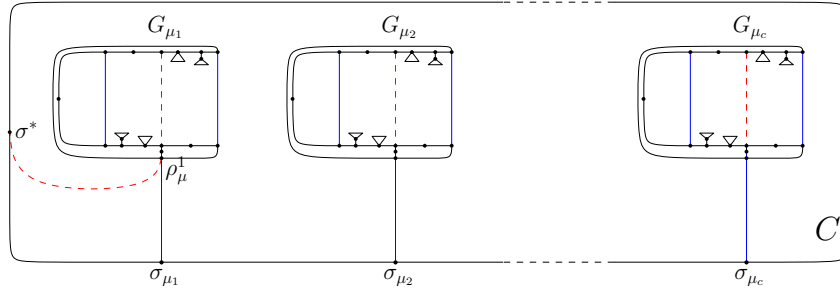


Figure 9: Composing all the cluster gadgets so that they lie in the same side of the frame cycle C .

To complete the construction of $\langle G_1, G_2 \rangle$, for each inter-cluster edge e we add to E_1 an edge $(y_\mu(e), y_\nu(e))$, where μ and ν are the clusters edge e is incident to.

Clearly, $\langle G_1, G_2 \rangle$ can be constructed in linear time and, hence, its size is linear in the size of $\langle \mathcal{C}(G, \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$. We now prove the equivalence.

(\Rightarrow) Suppose that $\langle G_1, G_2 \rangle$ admits a SEFE $\langle \Gamma_1, \Gamma_2 \rangle$. We show how to construct a c-planar drawing Γ of $\mathcal{C}(G, \mathcal{T})$ that is consistent with $\langle \mathcal{X}, \mathcal{Y} \rangle$.

In the following we will assume that the frame cycle C bounds the outer face of both Γ_1 and Γ_2 . This is not a loss of generality; in fact, since $G_\cup \setminus C$ is connected, where $G_\cup = (V, E_1 \cup E_2)$, all the vertices of G_1 and G_2 not in C lie on the same side of C in $\langle \Gamma_1, \Gamma_2 \rangle$, thus C delimits a face in both Γ_1 and Γ_2 , which we can assume to be the outer face.

We now prove a set of properties of $\langle \Gamma_1, \Gamma_2 \rangle$ with respect to the *vertex-cycle containment* relationship, that is, we prove that certain vertices have to lie in the interior or in the exterior of certain cycles of G_1 or G_2 in Γ_1 or Γ_2 , respectively.

We first focus on vertices and cycles belonging to the same cluster gadget G_μ .

For them, we first prove relationships involving cycles belonging to G_\cap , which hence hold in both Γ_1 and Γ_2 . First, the center c_μ of the wheel W_μ of G_μ lies in the interior of the rim of W_μ in both Γ_1 and Γ_2 , since vertex ρ_μ^1 is connected in G_2 with vertex σ_μ of C , which delimits the outer face of both Γ_1 and Γ_2 , by assumption. Also, since the subgraph of G_\cup induced by the vertices in $G_\mu \setminus W_\mu$ is connected, all the vertices of $G_\mu \setminus W_\mu$ lie in the exterior of the rim of W_μ in both Γ_1 and Γ_2 .

We then describe further relationships in G_μ only holding in Γ_2 : (i) all the vertices of the copies of the components c_i of μ lie in the interior of cycle $(\rho_\mu^4, \rho_\mu^5, \gamma_\mu)$ in Γ_2 , since they are all connected to γ_μ by means of paths in G_2 , and since they cannot lie in the interior of W_μ ; and (ii) all the other vertices of G_μ lie in the interior or on the boundary of cycle $(\rho_\mu^4, \beta_\mu, \alpha_\mu, \rho_\mu^3, \delta_\mu, \rho_\mu^2, \rho_\mu^7, \beta'_\mu, \alpha'_\mu, \rho_\mu^6, \delta'_\mu, \rho_\mu^5)$ in Γ_2 , since they are all connected to

vertices $\alpha_\mu, \alpha'_\mu, \beta_\mu$, and β'_μ by means of paths in G_2 , and since they cannot lie in the interior of W_μ .

Finally, we describe analogous relationships in G_μ only holding in Γ_1 : (i) all the vertices of the copies of the components c_i of μ , all the vertices of tree X_μ , and all the vertices of stars A_μ and B_μ lie in the interior or on the boundary of cycle $(\rho_\mu^4, \beta_\mu, \alpha_\mu, \rho_\mu^3, \rho_\mu^6, \delta'_\mu, \rho_\mu^5, \gamma_\mu)$ in Γ_1 , since they are all connected to vertices δ'_μ, α_μ , and β_μ by means of paths in G_1 , and since they cannot lie in the interior of W_μ ; (ii) all the other vertices of G_μ lie in the exterior or on the boundary of cycle $(\rho_\mu^2, \delta_\mu, \rho_\mu^3, \rho_\mu^6, \alpha'_\mu, \beta'_\mu, \rho_\mu^7, \rho_\mu^1)$ in Γ_1 , since they are all connected to vertices α'_μ, β'_μ , and δ_μ in G_1 , and since they cannot lie in the interior of W_μ .

We now consider vertex-cycle containment relationships between vertices not belonging to G_μ and cycles in G_μ . In particular, we prove that no vertex $v \notin G_\mu$ lies in the interior of a cycle of G_μ in both Γ_1 and Γ_2 . Namely, consider any vertex $v \in \nu$, where $\nu \neq \mu$ is a cluster of \mathcal{T} . Vertex v does not lie in the interior of cycle $C_\mu^2 = (\rho_\mu^2, \rho_\mu^7, \rho_\mu^1)$ in (Γ_1, Γ_2) , due to the fact that C_μ^2 is composed of edges belonging to G_2 , to the fact that ρ_ν^1 does not lie in the interior of C_μ^2 (since it is connected to σ_ν in G_2 , which is incident to the outer face), and to the fact that there exists a path in G_2 between v and ρ_ν^1 that does not contain vertices of C_μ^2 . Analogously, vertex v does not lie in the interior of cycle $C_\mu^1 = (\rho_\mu^2, \delta_\mu, \rho_\mu^3, \rho_\mu^6, \alpha'_\mu, \beta'_\mu, \rho_\mu^7, \rho_\mu^1)$ in (Γ_1, Γ_2) . In fact, there exists a path in G_1 not containing vertices of C_μ^1 between v and a leaf of Y_μ , which lies in the exterior of C_μ^1 . This path exists since the clusters-adjacency graph G_A is connected.

We remark that the latter consideration allows us to assume that edge (ρ_μ^3, ρ_μ^6) does not cross edge (ρ_μ^2, ρ_μ^7) in (Γ_1, Γ_2) . In fact, in this case we could remove such crossings by redrawing the portions of (ρ_μ^3, ρ_μ^6) lying outside cycle C_μ^2 so that edge (ρ_μ^3, ρ_μ^6) is drawn entirely inside C_μ^2 , without changing the vertex-cycle containment relationships between any vertex and cycle C_μ^1 . This implies that no edge of G_1 either between two vertices of G_μ or between two vertices of G_ν , for any cluster $\nu \neq \mu$, crosses edge (ρ_μ^2, ρ_μ^7) . In fact, any edge of G_1 between two vertices of G_μ lies entirely in the interior of cycle C_μ^1 , and thus of cycle C_μ^2 , while any edge of G_1 between two vertices of G_ν lies in the interior of cycle C_ν^1 , thus of cycle C_ν^2 , and hence entirely in the exterior of C_μ^2 .

We now show that we can further assume that all the other edges of G_1 cross edge (ρ_μ^2, ρ_μ^7) at most once. Note that the only remaining edges of G_1 that we have to consider are $(\sigma^*, \rho_{\mu_1}^1)$ and all the edges $(y_\nu(e), y_\eta(e))$ between a leaf of Y_ν in G_ν and a leaf of Y_η in G_η , where $\nu, \eta \in \mathcal{T}$ (possibly $\nu = \mu$). Namely, from the vertex-cycle containment relationships we proved above it follows that none of the vertices not belonging to G_μ lies inside cycle C_μ^2 , and that the only vertices of G_μ lying in the interior of C_μ^2 and not in the interior of C_μ^1 are the vertices of A'_μ , of B'_μ , and of Y_μ . Hence, if an edge e_r of G_1 crosses edge (ρ_μ^2, ρ_μ^7) more than once, these vertices are the only ones that might be enclosed in a region delimited by e_r and by (ρ_μ^2, ρ_μ^7) . However, this is not possible since all of them are connected to vertices of W_μ (namely α'_μ, β'_μ , and δ_μ) by means of paths of edges belonging to G_1 . Hence, any of these regions does not contain any vertex, and thus we can redraw edge e_r so that it crosses (ρ_μ^2, ρ_μ^7) at most once without changing the vertex-cycle containment relationships between any vertex and any cycle in G_1 .

In the following we will hence assume that, for every cluster $\mu \in \mathcal{T}$, edge (ρ_μ^2, ρ_μ^7) is crossed at most once by any edge of G_1 . In particular, this edge is crossed only by each edge $(y_\mu(e), y_\eta(e))$, incident to a leaf of tree Y_μ , which corresponds to an inter-cluster edge e of $\mathcal{C}(G, \mathcal{T})$ incident to μ .

We now show how to construct Γ . We denote by $\Theta(T_\mu)$, for each tree $T_\mu \in \{A_\mu, B_\mu, A'_\mu, B'_\mu, X_\mu, Y_\mu\}$, the order of the leaves in T in a clockwise Eulerian tour of T_μ starting from the leaf corresponding to e_μ in (Γ_1, Γ_2) . Further, we denote by $\Phi(Y_\mu)$ the order $\Theta(Y_\mu)$ restricted to the leaves corresponding to edges that are incident to multi-edge components of μ . Also, we will denote by $\overline{\Theta(T_\mu)}$ the reverse of order $\Theta(T_\mu)$ and by $\overline{\Phi(Y_\mu)}$ the reverse of order $\Phi(Y_\mu)$.

For each cluster $\mu \in \mathcal{T}$, the drawing of each multi-edge component c_i of μ in Γ coincides with the drawing in (Γ_1, Γ_2) of the copy of c_i in gadget G_μ , which belongs to G_\square . Also, the boundary $B(\mu)$ of the region $R(\mu)$ representing cluster μ coincides with the drawing of cycle C_μ^2 in Γ_2 .

We show how to draw the inter-cluster edges of $\mathcal{C}(G, \mathcal{T})$. In order to do that, we first construct a set Λ_μ of curves for each cluster $\mu \in \mathcal{T}$. Set Λ_μ contains a curve $\lambda_\mu(e)$ connecting $x_\mu(e)$ with $y_\mu(e)$, for each inter-cluster edge e incident to a multi-edge component of μ . The curves in Λ_μ are drawn as simple curves in the interior of cycle C_μ^2 so that (i) they do not cross each other, (ii) they do not cross any of the curves representing edges $(v, x_\mu(e'))$ and edges $(y_\mu(e'), y_\nu(e'))$, for every vertex $v \in \mu$ and for every inter-cluster edge e' incident to μ , and (iii) they do not cross any of the edges of G_\square between two vertices of the copy of a component c_i belonging to μ . This is always possible, since $\Theta(X_\mu) = \overline{\Phi(Y_\mu)}$, where we set $x_\mu(e) = y_\mu(e)$. We give a proof of this claim. First, the matching in E_2 between the leaves of X_μ and those of B_μ ensures that $\Theta(X_\mu) = \overline{\Theta(B_\mu)}$. Analogously, the matching in E_1 between the leaves of B_μ and those of A_μ ensures that $\overline{\Theta(B_\mu)} = \Theta(A_\mu)$. By repeating this argument while considering matchings in either E_2 or E_1 between the leaves of A_μ and of A'_μ , the leaves of A'_μ and of B'_μ , and the leaves of B'_μ and the leaves of Y_μ corresponding to inter-cluster edges incident to a multi-edge component of μ , we have that $\Theta(X_\mu) = \overline{\Theta(B_\mu)} = \Theta(A_\mu) = \overline{\Theta(A'_\mu)} = \Theta(B'_\mu) = \overline{\Phi(Y_\mu)}$.

We now draw each inter-cluster edge $e = (u, v)$ in Γ , where $u \in \mu$ and $v \in \nu$. If e is incident to a multi-edge component of μ and to a multi-edge component of ν , it is drawn as a composition of five parts. The first and the

last parts of e coincide with the drawing of edge $(u, x_\mu(e)) \in E_1$ of G_μ and of edge $(v, x_\nu(e)) \in E_1$ of G_ν in Γ_1 , respectively. The second and the fourth part coincide with curves $\lambda_\mu(e) \in \Lambda_\mu$ and $\lambda_\nu(e) \in \Lambda_\nu$, respectively. Finally, the middle part coincides with the drawing of edge $(y_\mu(e), y_\nu(e)) \in E_1$ in Γ_1 . If e is incident to a single-edge component of μ (of ν), then the first and the second part (the fourth and the fifth part) are not drawn.

Finally, for each single-edge component c_i of μ , let $e = (v, u)$ be the unique inter-cluster edge incident to c_i , with $v \in c_i$. We add to Γ a planar drawing of c_i in which v is incident to the outer face, so that v lies in the same position as $y_\mu(e)$ in (Γ_1, Γ_2) and there exists no crossing involving an edge of c_i .

We now prove that Γ is a c-planar drawing. Recall that we constructed region $R(\mu)$ for each cluster μ so that its boundary $B(\mu)$ coincides with C_μ^2 in Γ_2 . This implies that $R(\mu)$ contains all and only the vertices of μ , since all the vertices of the copies of the components c_i of μ , which belong to G_μ , lie inside C_μ^2 and since all the vertices of $\langle G_1, G_2 \rangle$ not in G_μ lie in the exterior of any cycle of G_2 .

Also, there exists no region-region crossings in Γ , since Γ_2 is a planar drawing of G_2 , and since C_μ^2 and C_ν^2 are vertex disjoint cycles in G_2 , for each $\mu, \nu \in \mathcal{T}$.

Further, there exists no edge-region crossing in Γ . In fact, the only intersection between $B(\mu)$, for each cluster $\mu \in \mathcal{T}$, and an edge of G is on the portion of $B(\mu)$ corresponding to edge (ρ_μ^2, ρ_μ^7) , since the remaining portion of $B(\mu)$ corresponds to edges in G_\cap , which are not crossed in $\langle \Gamma_1, \Gamma_2 \rangle$. Also, edge (ρ_μ^2, ρ_μ^7) is only crossed (once) by edges in G_1 between a leaf of Y_μ and a leaf of Y_ν , for some $\nu \in \mathcal{T}$. Hence, for each inter-cluster edge e incident to μ , only one of the five curves that have been used to draw e crosses $B(\mu)$, namely the middle one, and hence every edge of G crosses $B(\mu)$ at most once.

Finally, there exists no edge-edge crossing in Γ . Namely, observe that each edge e in G is either represented by an edge in G_1 (if e is an intra-cluster edge) or by the composition of edges in G_1 and curves in Λ_μ and Λ_ν , where μ and ν are the clusters e is incident to (if e is an inter-cluster edge). Hence, the planarity of the drawing of G in Γ descends from the planarity of Γ_1 and from the construction of the sets Λ_μ and Λ_ν .

We finally prove that Γ is consistent with $\langle \mathcal{X}, \mathcal{Y} \rangle$. Since the only edge of C_μ^2 that is crossed in $\langle \Gamma_1, \Gamma_2 \rangle$ is (ρ_μ^2, ρ_μ^7) , the linear order \mathcal{O}_μ in which the edges incident to μ cross $B(\mu)$ in Γ , starting from e_μ , coincides with the linear order in which the edges in G_1 cross (ρ_μ^2, ρ_μ^7) in $\langle \Gamma_1, \Gamma_2 \rangle$, starting from e_μ . By the planarity of Γ_1 , this order coincides with the reverse of $\Theta(Y_\mu)$, when we set $e = y_\mu(e)$. Hence, for every internal node ν of Y_μ , all the leaves of the subtree of Y_μ rooted at ν appear consecutively in \mathcal{O}_μ , and thus Γ is consistent with Y_μ . Analogously, Γ is consistent with X_μ , since $\Phi(Y_\mu) = \overline{\Theta(X_\mu)}$. Repeating this argument for all the clusters $\mu \in \mathcal{T}$ proves the statement.

(\Leftarrow) Suppose that $\mathcal{C}(G, \mathcal{T})$ admits a c-planar drawing Γ that is consistent with $\langle \mathcal{X}, \mathcal{Y} \rangle$. We show how to construct a SEFE $\langle \Gamma_1, \Gamma_2 \rangle$ of $\langle G_1, G_2 \rangle$. By Theorem 2, we can describe $\langle \Gamma_1, \Gamma_2 \rangle$ by means of the embeddings \mathcal{E}_1 and \mathcal{E}_2 of G_1 and of G_2 , respectively.

In the following we assume that edge e_{μ_1} (and hence vertex $\rho_{\mu_1}^1$) is incident to the outer face of the drawing of G in Γ . This is possible since e_{μ_1} is an inter-cluster edge.

We construct \mathcal{E}_1 and \mathcal{E}_2 in such a way that cycle C bounds a face, which we assume to be the outer face in both \mathcal{E}_1 and \mathcal{E}_2 . Clearly, this uniquely determines the rotation scheme of σ_μ and ρ_μ^1 in \mathcal{E}_2 , for each cluster $\mu \in \mathcal{T}$, and of σ^* and $\rho_{\mu_1}^1$ in \mathcal{E}_1 . Further, this implies that wheel W_μ , for each $\mu \in \mathcal{T}$, must be embedded so that c_μ lies in the interior of its rim in both \mathcal{E}_1 and \mathcal{E}_2 . We will embed all the other vertices in V and edges in E_1 and E_2 so that they lie in the exterior of the rim of each W_μ in both \mathcal{E}_1 and \mathcal{E}_2 . This uniquely determines the rotation scheme of all vertices $\rho_\mu^2, \dots, \rho_\mu^7$ in \mathcal{E}_2 and \mathcal{E}_1 .

Let \mathcal{O}_μ be the clockwise linear order in which the inter-cluster edges incident to μ cross $B(\mu)$ in Γ , starting from e_μ . We set the rotation scheme of the other vertices of G_μ so that: 1. $\overline{\Theta(Y_\mu)}$ coincides with \mathcal{O}_μ in both \mathcal{E}_1 and in \mathcal{E}_2 ; 2. the clockwise order of the paths connecting the center of star B'_μ with the leaves of tree Y_μ in G_2 not passing through β'_μ coincides with $\overline{\Theta(Y_\mu)}$ in \mathcal{E}_2 , when we identify each path with the leaf of Y_μ it is incident to; 3. $\Theta(B'_\mu)$ coincides with $\overline{\Phi(Y_\mu)}$ in \mathcal{E}_2 ; 4. each of $\overline{\Theta(A'_\mu)}, \overline{\Theta(A_\mu)}, \overline{\Theta(B_\mu)}$, and $\overline{\Theta(X_\mu)}$ coincides with $\overline{\Phi(Y_\mu)}$ in both \mathcal{E}_1 and in \mathcal{E}_2 ; 5. each vertex v of the copy of a multi-edge component c_i of μ has the same rotation scheme in \mathcal{E}_1 as the corresponding vertex in Γ , where we replace e with $(v, x_\mu(e))$, if e is an inter-cluster edge incident to μ ; 6. each vertex v of the copy of a multi-edge component c_i of μ has the same rotation scheme in \mathcal{E}_2 as the corresponding vertex in Γ , where we remove all of the inter-cluster edges incident to v , except for one edge e_v , which we replace with $(v, z(c_i))$; 7. for each vertex $z(c_i)$ of C_μ , the order of the edges in the rotation scheme of $z(c_i)$ in \mathcal{E}_2 is the same as the order in which these edges appear in a counter-clockwise walk around the boundary of c_i in Γ , where we remove all of the inter-cluster edges incident to v , except for edge e_v ; 8. the center of C_μ has any rotation scheme in both \mathcal{E}_1 and in \mathcal{E}_2 ; 9. edges $(v, x_\mu(e_\mu))$, $(\delta'_\mu, x_\mu(e_\mu))$, and $(\rho_\mu, x_\mu(e_\mu))$ appear in this order in the rotation scheme of $x_\mu(e_\mu)$ in \mathcal{E}_1 , where v is the vertex of μ edge e_μ is incident to; 10. edges $(b_\mu(e_\mu), x_\mu(e_\mu))$, $(\delta'_\mu, x_\mu(e_\mu))$, and $(\rho_\mu, x_\mu(e_\mu))$ appear in this order in the rotation scheme of $x_\mu(e_\mu)$ in \mathcal{E}_2 ; 11. edges $(\xi_\mu, y_\mu(e_\mu))$, $(\delta_\mu, y_\mu(e_\mu))$, and $(y_\nu(e_\mu), y_\mu(e_\mu))$ appear in this order in the rotation scheme of $y_\mu(e_\mu)$ in \mathcal{E}_1 , where ν is the other

cluster to which e_μ is incident; 12. edges $(\xi_\mu, y_\mu(e_\mu))$, $(\delta_\mu, y_\mu(e_\mu))$, and $(b'_\mu(e_\mu), y_\mu(e_\mu))$ appear in this order in the rotation scheme of $y_\mu(e_\mu)$ in \mathcal{E}_2 . Note that the rotation scheme of the remaining vertices of G_μ in \mathcal{E}_1 and \mathcal{E}_2 (namely the leaves of stars C_μ , B_μ , A_μ , B'_μ , and A'_μ , and the leaves of trees X_μ and Y_μ different from $x_\mu(e_\mu)$ and from $y_\mu(e_\mu)$, respectively) is unique, since they have degree less or equal 2 in G_1 and G_2 .

We prove that both \mathcal{E}_1 and \mathcal{E}_2 are planar. First note that there exists a planar embedding of X_μ and of Y_μ so that $\overline{\Theta(Y_\mu)} = \mathcal{O}_\mu$ and $\Theta(X_\mu) = \overline{\Phi(Y_\mu)}$, since Γ is consistent with X_μ and with Y_μ . The embedding of the biconnected components of G_2 induced by the vertices (i) of X_μ and of B_μ , (ii) of A_μ and of A'_μ , and (iii) of B'_μ and of Y_μ are planar since $\Theta(X_\mu) = \overline{\Theta(B_\mu)}$, since $\Theta(A_\mu) = \overline{\Theta(A'_\mu)}$, and since the clockwise order of the paths connecting the center of star B'_μ with the leaves of tree Y_μ in G_2 not passing through β'_μ coincides with $\overline{\Theta(Y_\mu)}$ in \mathcal{E}_2 .

Analogously, the embedding of the biconnected components of G_1 induced by the vertices (i) of A_μ and of B_μ , and (ii) of A'_μ and of B'_μ , are planar since $\Theta(A_\mu) = \overline{\Theta(B_\mu)}$, and since $\Theta(B'_\mu) = \overline{\Theta(A'_\mu)}$, respectively.

Also, the embedding of the biconnected component of G_2 composed of the copy of each multi-edge component c_i of μ , of vertex $z(c_i)$, and of the edges between them is planar, by the construction of the rotation scheme of $z(c_i)$. Further, the embedding of the subgraph of G_1 composed of the copies of all the multi-edge components c_i of μ , of tree X_μ , and of the edges between them is planar since $\Theta(X_\mu)$ coincides with \mathcal{O}_μ restricted to the inter-cluster edges incident to the multi-edge components of μ . Finally, the embedding of the biconnected component of G_1 composed of tree Y_μ , of tree Y_ν , and of the edges between their leaves, for each two adjacent clusters μ and ν in G_A , is planar since $\overline{\Theta(Y_\mu)} = \Theta(Y_\nu)$, restricted to the inter-cluster edges incident to both μ and ν . This is due to the fact that $\mathcal{O}_\mu = \Theta(Y_\mu)$, that $\mathcal{O}_\nu = \Theta(Y_\nu)$, and that \mathcal{O}_μ coincides with the reverse of \mathcal{O}_ν , when both orders are restricted to the edges incident to both μ and ν , by the c-planarity of Γ . Note that, since Γ has edge e_{μ_1} on the outer face, vertex $\rho_{\mu_1}^1$ is not enclosed by any cycle of G_1 , except for C . Hence, vertices $\rho_{\mu_1}^1$ and σ^* are incident to the same face of \mathcal{E}_1 .

The planarity of \mathcal{E}_1 and of \mathcal{E}_2 , restricted to the edges of G_μ in E_1 and in E_2 , respectively, is implied by the planarity of the embedding of each of the above considered components of G_1 and G_2 , and by the order in which δ_μ , α_μ , β_μ , γ_μ , δ'_μ , α'_μ , and β'_μ appear along the rim of W_μ .

Further, since each G_μ is only connected to the frame cycle C via edge $(\rho_{\mu_1}^1, \sigma_\mu)$, the planarity of \mathcal{E}_2 restricted to the edges of each G_μ in E_2 implies the planarity of the whole \mathcal{E}_2 . To complete the proof of the planarity of \mathcal{E}_1 , it only remains to consider the embedding of the subgraph of G_1 induced by the vertices of all trees Y_μ , with $\mu \in \mathcal{T}$. The planarity of this subgraph descends from the planarity of the embedding of the subgraph of G_1 induced by the vertices of each two trees Y_μ and Y_ν such that μ and ν are adjacent in G_A , from the fact that Γ is consistent with Y_μ , for each $\mu \in \mathcal{T}$, and from the c-planarity of Γ . This completes the proof that $\langle \Gamma_1, \Gamma_2 \rangle$ is a SEFE of $\langle G_1, G_2 \rangle$.

We conclude the lemma by proving that $\langle G_1, G_2 \rangle$ can be transformed into an equivalent instance in which $G_\cap = (V, E_1 \cap E_2, G_1, \text{ and } G_2)$ satisfy the required properties.

We note that G_1 is connected, since G_A is connected. Also, G_1 contains cut-vertices σ^* and $\rho_{\mu_1}^1$. Further, for each cluster $\mu \in \mathcal{T}$, G_1 contains cut-vertices δ_μ , $y_\mu(e_\mu)$, δ'_μ , $x_\mu(e_\mu)$, γ_μ , the center of star C_μ , the internal vertices of X_μ , and possibly the internal vertices of Y_μ . We now show that all these cut-vertices are incident to at most two non-trivial blocks of G_1 . Namely, vertices σ^* , $\rho_{\mu_1}^1$, and vertices δ_μ , δ'_μ , γ_μ , $x_\mu(e_\mu)$, and $y_\mu(e_\mu)$, for each cluster $\mu \in \mathcal{T}$, are incident to exactly two blocks in G_1 . The center of star C_μ is incident only to non-trivial blocks. Each internal vertex c_i of X_μ is incident to at most one non-trivial block, that is, the block composed of vertex c_i , of the leaves of X_μ incident to c_i , and of the vertices of the copy of the multi-edge component c_i in G_μ . Each internal vertex ν of Y_μ is incident to at most one non-trivial block, that is, the one composed of ν , of the leaves of Y_μ incident to ν , of the vertex μ in Y_ν , and of the leaves of Y_ν incident to μ .

We note that G_2 is connected, by construction. Also, for each cluster $\mu \in \mathcal{T}$, graph G_2 contains cut-vertices γ_μ , the center of star C_μ , and vertices $z(c_i)$, for each component c_i of μ . We now show that all these cut-vertices are incident to at most two non-trivial blocks of G_2 . Namely, vertex γ_μ and vertices $z(c_i)$, for each multi-edge component c_i of μ , are incident to exactly two blocks, while the center of C_μ is incident only to non-trivial blocks.

Finally, no vertex of a copy of a multi-edge component c_i of μ is a cut-vertex in either G_1 or G_2 . This is due to the fact that, by assumption, every block of c_i that is a leaf in the BC-tree of c_i has at least an inter-cluster edge incident to one of its vertices that is not a cut-vertex of c_i . Hence, all the vertices of the copy of c_i , together with the vertex $c_i \in X_\mu$ and with the leaves of X_μ incident to it, belong to the same block of G_1 . Also, all the vertices of the copy of c_i , together with vertex $z(c_i)$, belong to the same block of G_2 .

By Claim 2, the cycles of G_\cap can be removed so that G_\cap becomes a forest, without altering the properties of $\langle G_1, G_2 \rangle$. Observe that the only cycles contained in G_\cap are the frame cycle C , the cycles in W_μ , for each $\mu \in \mathcal{T}$, and (possibly) the cycles in the copy of some multi-edge component c_i of G_ν , for some cluster $\nu \in \mathcal{T}$. This concludes the proof of the lemma. \square

We conclude by exploiting Lemma 7 to prove the main result of the section.

Theorem 6. An instance $\langle \mathcal{C}(G(V, E), \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ of INCLUSION-CONSTRAINED C-PLANARITY can be tested in $O(|V|^2)$ time.

Proof. Since for each inter-cluster edge e there exist at most two components trees in \mathcal{X} and exactly two neighbor-clusters trees in \mathcal{Y} with a leaf corresponding to e , we have $|\mathcal{X}|, |\mathcal{Y}| \in O(|E|)$. Also, since G is planar, $|E| \in O(|V|)$, and $|G| \in O(|V|)$. Hence, $s = |G| + |\mathcal{X}| + |\mathcal{Y}| \in O(|V|)$.

Apply Lemma 7 to $\langle \mathcal{C}(G(V, E), \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ to construct in $O(s)$ time an equivalent instance of SEFE satisfying the conditions of Theorem 1, which can be tested in $O(s^2) \in O(|V|^2)$ time. \square

7 Conclusions and Open Problems

In this paper we studied the problem of constructing c-planar drawings with pipes of flat c-graphs. We presented algorithms to test the existence of such drawings when the number of certain components is small, in different scenarios, namely when the clusters-adjacency graph is a path (STRIP PLANARITY), when it has a fixed embedding (C-PLANARITY WITH EMBEDDED PIPES), and when it has no restrictions (C-PLANARITY WITH PIPES).

Several questions are left open. We find particularly interesting to determine whether there exist combinatorial properties of the nesting of the components that would allow us to reduce the number of possible components trees, analogous to the ones we could prove for the pipe-neighbor-clusters trees. We remark that the introduction of the components trees already allowed us to make the running time of our algorithms, and in particular of the FPT algorithm, independent of the size of each component.

Another natural question concerns the possibility of extending our results to problem C-PLANARITY. An important goal would be to determine the complexity of this problem for flat c-graphs in the case in which each cluster contains at most two components. Efficient algorithms for this case exist only when the underlying graph has a fixed embedding [15], when also each co-cluster has at most two components [6], or when the cut-vertices of the clusters-adjacency graph have at most two non-trivial blocks [6].

We would like to point out that this latter result is obtained by considering a graph that is in fact the one we defined as clusters-adjacency graph. Namely, the authors of [6] introduced a data structure, called CD-tree, which is a star when the considered c-graph $\mathcal{C}(G, \mathcal{T})$ is flat; in this case, the skeleton associated to the central node of this star turns out to coincide with the clusters-adjacency graph G_A of $\mathcal{C}(G, \mathcal{T})$. In this paper [6], problem C-PLANARITY for flat c-graphs is described in terms of a specific constrained-planarity problem for G_A , namely the problem of computing a planar embedding of this graph satisfying a set of partitioned PQ-constraints. The mentioned result for flat c-graphs is then obtained by showing that the given restrictions for the original c-graph allow to generate instances of this constrained-planarity problem that can be solved by means of the Simultaneous PQ-ordering framework [7]. The authors also extended their result to give an FPT algorithm for the same problem in two parameters that depend on the total number of clusters and on the number of edges leaving a cluster. We remark that analogous results (with slightly different parameters for the FPT algorithm) could be obtained using the techniques of our paper; a key property for this is the fact that, when G_A is biconnected, the neighbor-clusters tree of each cluster can be proved to be unique. We thus ask whether deeper considerations on the possible nesting configurations of the clusters could be used to further reduce the number of neighbor-clusters trees to be considered even when the cut-vertices of G_A have a larger number of non-trivial blocks.

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